

Physical Applications of Geometric Algebra

Part I Non-Relativistic Physics

Chris Doran and Anthony Lasenby

www.mrao.cam.ac.uk/~clifford/ptIIIcourse

Chapter 1

Introduction to Geometric Algebra

The ideas and concepts of physics are best expressed in the language of mathematics. But this language is far from unique. Many different algebraic systems exist and are in use today, all with their own advantages and disadvantages. In this course we will study the most powerful available mathematical system developed to date — Clifford's *geometric algebra*. This will be presented as a new tool to add to your existing base as either a theoretician or experimentalist. The aim will be to introduce new techniques via their applications, rather than as purely formal mathematics. These applications will be diverse, emphasising the generality and portability of geometric algebra. This will help to promote a more inter-disciplinary understanding of science.

This course is divided into 3 sections, looking at the applications of geometric algebra (GA) to non-relativistic physics, relativistic physics and gravitation respectively. During this course we will discover a new, powerful technique for handling rotations in arbitrary dimensions, and analyse the insights this brings to the mathematics of special and general relativity. We will also uncover the links between rotations, *bivectors* and the structure of the Lie groups which underpin much of modern physics.

In a separate development we will learn how to extend the concept of a complex analytic function in 2-d (*i.e.* a function satisfying the Cauchy-Riemann equations) to arbitrary dimensions, and show how this is applied in quantum theory and electromagnetism. This will enable us to unite all four Maxwell equations into a single equation ($\nabla F = J$), and develop new techniques for solving it. In the final section of the course we will combine these ideas to construct a gauge theory of gravitation in (flat) Minkowski spacetime, which is still consistent with General Relativity. Our new understanding of gravitation will enable us to quickly reach advanced applications such as black holes and cosmology. Throughout, the emphasis will be placed on the unity of the mathematics underpinning each of these topics.

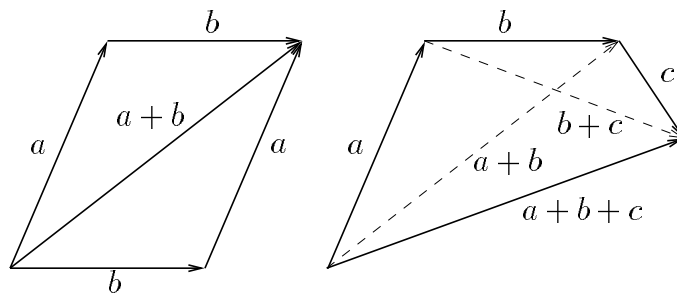


Figure 1.1: A *geometric picture of vector addition*. The result of $a + b$ is formed by adding the tail of b to the head of a . As is shown, the resultant vector $a + b$ is the same as $b + a$. This finds an algebraic expression in the statement that addition is commutative. In the right-hand diagram the vector $a + b + c$ is constructed two different ways, as $a + (b + c)$ and as $(a + b) + c$. The fact that the results are the same is the geometric expression of the associativity of vector addition.

1.1 Vector Spaces

We begin with a quick revision of the mathematical properties of vectors and the products that can be formed between them. This leads naturally to Clifford's *geometric product*, which forms the basis for most of this course. We should all be familiar with the basic idea of a *vector space*. These consist of a set of objects a, b , (the vectors) and a rule for adding them which is *commutative*

$$a + b = b + a, \quad (1.1)$$

and *associative*;

$$a + (b + c) = (a + b) + c. \quad (1.2)$$

This latter property enables us to write expressions such as $a + b + c$ without ambiguity. Vectors can often be visualised as directed line segments, and for these each of the two preceding properties have a clear geometric equivalent (see Fig. 1.1).

Vector spaces also contain a multiplication operation which allows vectors to be multiplied by scalars. For this course we will restrict entirely to real scalars, as complex structures will be seen to emerge geometrically. The following further properties hold for any scalars λ, μ and vectors a and b :

1. $\lambda(a + b) = \lambda a + \lambda b$;
2. $(\lambda + \mu)a = \lambda a + \mu a$;
3. $(\lambda\mu)a = \lambda(\mu a)$;

4. If $1\lambda = \lambda$ for all scalars λ then $1a = a$ for all vectors a .

These rules define a vector space completely. Note that the $+$ operation connecting scalars is different from the $+$ operation connecting the vectors. There is no ambiguity, however, in using the same symbol for both.

All (finite) vector spaces have an associated *dimension*, and a set of linearly independent vectors of this number are said to form a *basis*. Any vector in the space can be written uniquely as a sum of these basis vectors multiplied by scalar coefficients (the components). One can proceed to prove a surprising wealth of geometric results with just these axioms alone, but it is clear that they lack a key structure — a rule for multiplying vectors. In the following sections we look at various familiar products for vectors, prior to introducing Clifford's geometric product.

1.2 The Scalar Product

A vector space as defined above is currently insufficient for Euclidean geometry as it lacks the concepts of distance and angle. Without these we cannot define circles, or perpendicular lines. Both distances and angles can be defined through the introduction of a scalar product for vectors. This is usually written as $a \cdot b$ and returns a scalar. In Euclidean space the scalar product is positive definite,

$$a^2 = a \cdot a > 0 \quad \forall a \neq 0. \quad (1.3)$$

This enables us to define the length of a vector $|a|$ as

$$|a| = \sqrt{(a \cdot a)}. \quad (1.4)$$

From this definition we recover the Schwarz inequality

$$\begin{aligned} (a + \lambda b)^2 &\geq 0 && \forall \lambda \\ \implies a^2 + 2\lambda a \cdot b + \lambda^2 b^2 &\geq 0 && \forall \lambda \\ \implies (a \cdot b)^2 &\leq |a|^2 |b|^2. \end{aligned} \quad (1.5)$$

This is used to define the cosine of the angle between a and b via

$$a \cdot b = |a| |b| \cos(\theta). \quad (1.6)$$

Two vectors whose scalar product vanishes are said to be orthogonal. It is usually convenient to work with basis sets of vectors which are mutually orthogonal, and where all of the basis vectors have been normalised to 1. In spaces which are not positive definite, such as Minkowski spacetime, we can still construct an orthonormal frame, but now some vectors have square $+1$ and some have square -1 . The Schwarz inequality does not necessarily hold, unless the vectors in question inhabit a subspace which is positive definite.

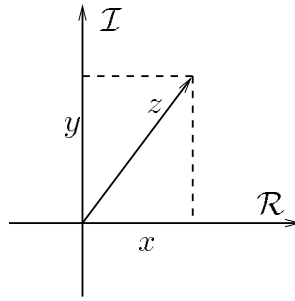


Figure 1.2: *The Argand Diagram*. Complex numbers can be used to represent vectors in 2-d, as well the operations of rotation and dilation applied to vectors.

1.3 Complex Numbers

The next product for vectors to consider is defined by complex numbers. Recall that any complex number can be taken as defining a point in the Argand diagram (Fig. 1.2). The product of two complex numbers can therefore be viewed as a way of multiplying together two vectors in 2-d. If we let $z = x + iy$ and form z^2 we see that the real term here does not have the desired form in terms of the square of the length. Instead a more useful product is defined by

$$zz^* = (x + iy)(x - iy) = x^2 + y^2, \quad (1.7)$$

which does return the square of the length of the vector. A product of two vectors in a plane, z and $w = u + vi$, can therefore be constructed as

$$zw^* = (x + iy)(u - iv) = xu + vy + i(uy - vx). \quad (1.8)$$

The real part of the right-hand side is simply the scalar product back again. To understand the imaginary term consider the polar representation

$$z = |z|e^{i\theta}, \quad w = |w|e^{i\phi} \quad (1.9)$$

so that

$$zw^* = |z||w|e^{i(\theta - \phi)}. \quad (1.10)$$

The imaginary term is $i|z||w|\sin(\theta - \phi)$, where $\theta - \phi$ is the angle between the two vectors. The magnitude of this term is therefore the area of the parallelogram defined by z and w . The sign of the term conveys information about the *handedness* of the area element swept out by the two vectors. This will be defined more carefully later.

We have a satisfactory interpretation for both the real and imaginary parts of the product zw^* . The surprising feature is that these are still both parts of a complex number. We thus have a second interpretation for complex addition, as a sum between

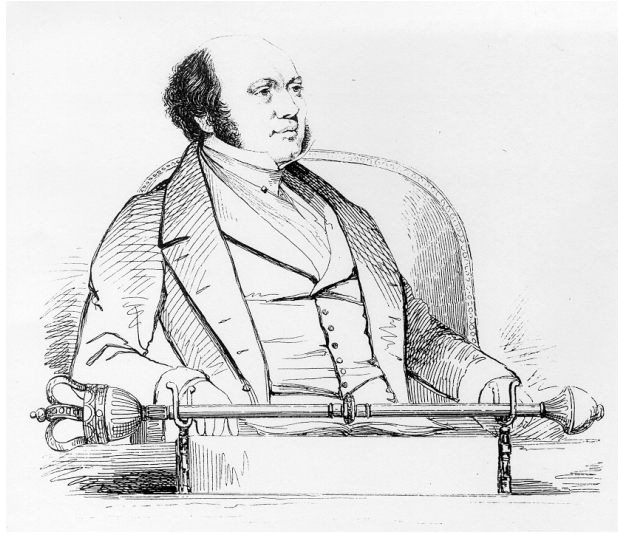


Figure 1.3: *William Rowan Hamilton 1805–1865*. Inventor of quaternions, and one of the key scientific figures of the 19th century.

scalar objects and objects representing plane segments. The advantages of adding these together are precisely the advantages of working with complex numbers as opposed to pairs of real numbers. This is a theme which runs throughout geometric algebra. But before defining this we need a brief detour through the quaternions and Grassmann's algebra, which provide the missing concept of an antisymmetric product.

1.4 Quaternions

A central problem being tackled in the first part of the 19th Century was how best to represent 3-d rotations. *W.R. Hamilton* (Fig. 1.3) worked on this for many years, looking for a way to generalise complex arithmetic to 3-d space, and in 1844 he finally produced his *quaternions*. These are indeed excellent for representing rotations, but the final algebra contains 4 objects, $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, instead of the expected 3. The quaternion algebra is defined by the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \quad (1.11)$$

which can be seen to define a *closed* algebra. The quaternions also form a *division* algebra, which means that every element has a unique inverse. Hamilton thought that this was important, but in fact it is not a defining property of a geometric algebra. (The quaternions are one of only four division algebras: the real numbers, complex numbers, quaternions and octonions).

The revolutionary new idea of the quaternion algebra is that the basis elements *anti-commute*. To check this we form

$$\mathbf{i}\mathbf{j} = -\mathbf{i}\mathbf{j}\mathbf{k}\mathbf{k} = \mathbf{k}, \quad (1.12)$$

and

$$\mathbf{j}\mathbf{i} = -\mathbf{j}\mathbf{i}\mathbf{j}\mathbf{k} = -\mathbf{k} = -\mathbf{i}\mathbf{j}. \quad (1.13)$$

The remaining products between the \mathbf{i} , \mathbf{j} and \mathbf{k} all follow this pattern. The idea of non-commuting products was only beginning to emerge during the 19th Century as mathematicians started to uncover the properties of matrices and groups.

The fact that the quaternion algebra contained 4 elements, whereas only 3 were needed to describe a vector in 3-d, caused considerable confusion. Hamilton's solution was to define a 'pure' quaternion as one containing no real part, and to treat these as vectors. This is clearly sensible, and is the origin of the traditional use of \mathbf{i} , \mathbf{j} , \mathbf{k} for unit vectors in 3-d. However, the negative norm was thought by some to be undesirable, and the correct way to rotate these vectors was unclear. We return to this topic in Section 2.5 when we rediscover the quaternion algebra in terms of the geometric algebra of 3-d space.

One feature of the quaternions which was immediately adopted by mathematicians and physicists is seen if we multiply together 2 pure quaternions (*i.e.* vectors). We set

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \quad (1.14)$$

and write their product as

$$\mathbf{a}\mathbf{b} = c_0 + \mathbf{c}. \quad (1.15)$$

The scalar term c_0 is minus the familiar scalar product of two vectors. The vector term is more interesting as it is given by

$$\mathbf{c} = (a_2b_3 - b_2a_3)\mathbf{i} + (a_3b_1 - b_3a_1)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k}. \quad (1.16)$$

The components of \mathbf{c} are those of the cross product of \mathbf{a} and \mathbf{b} . The cross product only exists in 3-d space and has the properties that $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane defined by \mathbf{a} and \mathbf{b} , with magnitude $ab \sin(\theta)$ and such that \mathbf{a} , \mathbf{b} and $\mathbf{a} \times \mathbf{b}$ form a right-handed set. (These geometric rules are sufficient to define the cross product uniquely).

The cross product was immediately recognised as being very useful in subjects such as mechanics and electromagnetic theory. Gibbs, for example, advocated forgetting about quaternions altogether and just keeping the separate scalar and cross products. From the modern perspective this is an extremely narrow viewpoint, but in the 19th Century this was an acceptable solution to the quaternion muddle and was widely adopted.



Figure 1.4: *Hermann Günther Grassmann (1809–1877)*. German mathematician and schoolteacher, famous for inventing the algebra which now bears his name.

1.5 The Outer Product

The cross product has one major failing — it only exists in 3 dimensions. In 2-d there is nowhere else to go, whereas in 4-d the concept of a vector orthogonal to a pair of vectors is not unique. To see this, consider 4 orthonormal vectors $e_1 \dots e_4$. If we take the pair e_1 and e_2 and attempt to find a vector perpendicular to both of these, we see that any combination of e_3 and e_4 will do.

What we need is a means of encoding a plane geometrically, without relying on the notion of a vector perpendicular to it. This was the problem solved by *H.G. Grassmann* (Fig. 1.4) in his *Ausdehnungslehre*. This was published in 1844, the same year that Hamilton introduced his quaternion algebra. Grassmann introduced an *outer* or *exterior* product for vectors, which we now write as $a \wedge b$ or ‘a wedge b’. Unlike the cross product, this results in a *new* algebraic object — a bivector — with the geometric interpretation of a plane. This is the directed area swept out by a and b and can be visualised as the parallelogram obtained by sweeping one vector along the other (Fig. 1.5). Changing the order of the vectors reverses the orientation of the plane. The plane has area $|a||b|\sin(\theta)$, which is defined to be the magnitude of $a \wedge b$.

The outer product of two vectors has the following properties:

1. The outer product of two vectors is antisymmetric,

$$a \wedge b = -b \wedge a, \tag{1.17}$$

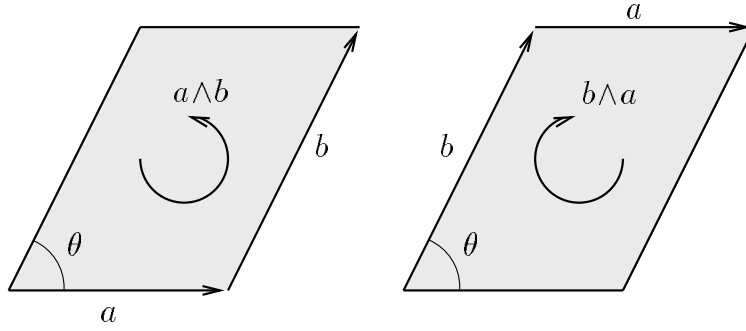


Figure 1.5: *The Outer Product.* The outer or wedge product of a and b returns a directed area element of area $|a||b|\sin(\theta)$.

which follows from the geometric definition. It follows that $a \wedge a = 0$.

2. Bivectors form a linear space, the same way that vectors do. In 3-d the addition of bivectors is easy to visualise (see Fig. 1.6). In higher dimensions this addition is not always so easy to visualise, because two planes need not share a common line. This can have some interesting consequences.
3. The outer product is distributive

$$a \wedge (b + c) = a \wedge b + a \wedge c. \quad (1.18)$$

This helps to visualise the addition of bivectors.

It is traditional to represent bivectors with parallelograms, but it is important to realise that bivectors do not specify a shape. This is clear from the fact that, if $a' = a + \lambda b$, we still have

$$a' \wedge b = a \wedge b + \lambda b \wedge b = a \wedge b. \quad (1.19)$$

So, given $a \wedge b$, there is no unique way to recover the vectors a and b . All that the bivector encodes is the plane itself, together with an area and a handedness. For this reason it is sometimes better to replace the directed parallelogram with a directed circle.

To understand the outer product of two vectors a and b further it is useful to introduce a pair of basis vectors for the plane defined by the vectors. If we denote these e_1 and e_2 we can write

$$a = a_1 e_1 + a_2 e_2, \quad b = b_1 e_1 + b_2 e_2. \quad (1.20)$$

The outer product of these is

$$\begin{aligned} (a_1 e_1 + a_2 e_2) \wedge (b_1 e_1 + b_2 e_2) &= a_1 b_2 e_1 \wedge e_2 + a_2 b_1 e_2 \wedge e_1 \\ &= (a_1 b_2 - a_2 b_1) e_1 \wedge e_2. \end{aligned} \quad (1.21)$$

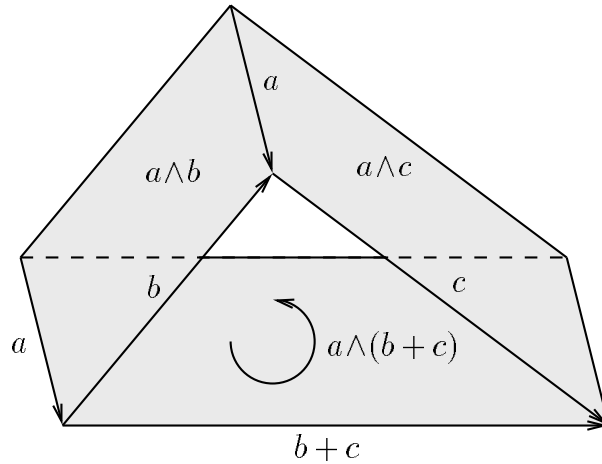


Figure 1.6: *Bivector Addition*. In 3-d bivector addition can be visualised like vector addition. The ‘tail’ of one bivector is added to the ‘head’ of the other.

The component of the bivector is precisely that found for the imaginary term in the complex product zw^* .

In general, the components of $a \wedge b$ are $a_i b_j$ where the $[\]$ denotes antisymmetrisation. Grassmann extended the definition of the exterior product to arbitrary numbers of vectors, which results in a very general algebraic tool. His work was largely ignored during his lifetime, due in part to his status and also to his writing style. His choice of notation and naming conventions was particularly awkward. Since his death, however, his work has given rise to the influential and fashionable areas of *differential forms* and *Grassmann (anticommuting) variables*. The latter are fundamental to the foundation of much of modern supersymmetry and superstring theory. We will not follow this development here, and will instead recover the properties of the outer product in terms of Clifford’s geometric product.

1.6 The Geometric Product

The next key step in our story was made in 1878 by the mathematician *W.K. Clifford* (Fig. 1.7). He appears to have been one of the select group of mathematicians at the time who had read and understood Grassmann’s work. Like Hamilton, Clifford was looking for an algebra of vectors which would extend the power of complex analysis to arbitrary dimensions. The key is the correct identification of the roles of the terms in the complex product zw^* . The scalar term is simply the scalar product of the vectors specified by z and w . The imaginary coefficient is the same as that of the outer product so, with a suitable interpretation, the imaginary term can be treated as the

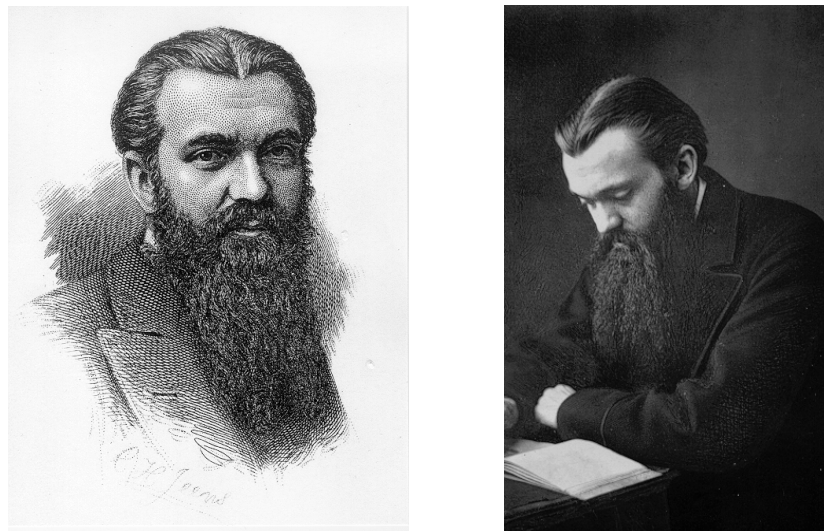


Figure 1.7: *William Kingdon Clifford 1845–1879*. Mathematician and philosopher. He died tragically young at the age of only 33, though he did have enough time to grow his prodigious beard.

outer product. What the complex arithmetic is telling us to do then is to combine these two terms into a single product. Clifford's idea was simply to *define* a product with precisely these properties. This is the *geometric product*, written simply as ab , and satisfying

$$ab = a \cdot b + a \wedge b. \quad (1.22)$$

The right-hand side is a sum of two distinct objects — a scalar and a bivector. This looks strange, but should be thought of in precisely the way we think of a complex number, formed from the sum of real and imaginary parts. These are carried round in a single entity, which provides for many mathematical simplifications.

From the symmetry and antisymmetry of the scalar (or *inner*) and outer products respectively, we see that

$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b. \quad (1.23)$$

It follows that

$$a \cdot b = \frac{1}{2}(ab + ba) \quad a \wedge b = \frac{1}{2}(ab - ba). \quad (1.24)$$

We can thus *define* both the inner and outer products in terms of the geometric product. This suggests that we should treat the geometric product as the primitive one for the axiomatic development of the subject, which is what Clifford did. He extended the geometric product to arbitrary numbers of vectors with the following rules:

1. The geometric product is associative,

$$a(bc) = (ab)c = abc. \quad (1.25)$$

2. The geometric product is distributive over addition

$$a(b + c) = ab + ac. \quad (1.26)$$

3. The symmetric part of the geometric product of two vectors is a scalar. An alternative way to say this is that the square of any vector is a scalar. To see why, consider the expansion

$$\begin{aligned} (a + b)^2 &= (a + b)(a + b) \\ &= a^2 + b^2 + ab + ba. \end{aligned} \quad (1.27)$$

It follows that

$$ab + ba = (a + b)^2 - a^2 - b^2. \quad (1.28)$$

So, if the left-hand side is a scalar then so is the right-hand side, and vice-versa.

1.7 History of Clifford Algebra

The algebra formed by a set of vectors multiplied by the geometric product is called a *geometric algebra*, often also called a *Clifford algebra*. The product is associative, like Grassmann's outer product, but has the crucial extra feature of being *invertible*, like Hamilton's quaternion product. Indeed, Clifford's original motivation was to unite Grassmann's and Hamilton's algebras into a single structure. In Clifford's geometric algebra an equation of the type $ab = C$ had the solution $b = a^{-1}C$. Neither the scalar or exterior products are capable of this inversion on their own.

Clifford's system combined all of the advantages of quaternions with those of vector geometry, so geometric algebra should have then gone forward as the main system for mathematical physics. However, two events conspired against this. The first was Clifford's untimely death at the age of just 33 and at the height of his powers. The second was *Gibbs'* advocacy of the familiar vector calculus. This was well suited to the theory of electromagnetism as it stood at the end of the 19th century, and Gibbs' considerable reputation meant that this system eclipsed Clifford and Grassmann's work. By the time special relativity arrived, and physicists realised that they needed a system capable of handling 4-d space, the crucial insights of Grassmann and Clifford had been lost to a generation.

In the 1920's Clifford algebra resurfaced as the algebra underlying *quantum spin*. In particular the algebra of the *Pauli* and *Dirac* spin matrices became indispensable in quantum theory. However, these were treated just as algebras — the *geometrical* meaning was lost. For this reason we still employ the term 'Clifford algebras' when the algebra is used solely for its formal algebraic properties. When applied in its proper, geometric setting however, we prefer to use Clifford's own name of *geometric algebra*.

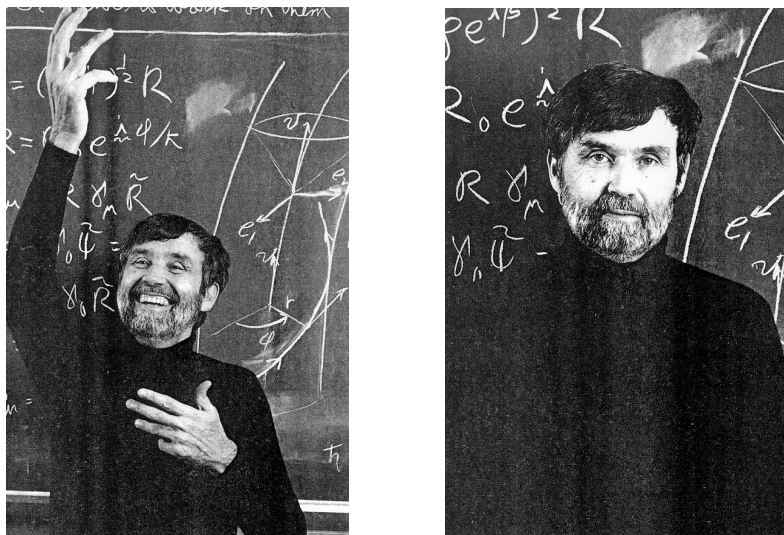


Figure 1.8: *David Orlin Hestenes*. Inventor of geometric calculus and first to draw attention to the universal nature of geometric algebra. He wrote the influential *Space-time Algebra* in 1966, and followed this with a fully developed formalism in *Clifford Algebra to Geometric Calculus* (D. Hestenes & G. Sobczyk, 1984). This was followed by the (much easier!) *New Foundations for Classical Mechanics* in 1986.

This neatly avoids the minor historical point that Grassmann in his later work was actually the first to write down a geometric (Clifford) product!

The situation remained largely unchanged until the 1960's, when *David Hestenes* (Fig. 1.8) started to recover the geometrical meaning underlying the Pauli and Dirac algebras. His original motivation was to gain some insight into the nature of quantum mechanics, but he soon realised that, properly applied, Clifford's system was nothing less than a universal language for mathematics, physics and engineering! It has taken Hestenes many years to convince people of this fact, but interest is now gathering pace. Part of the original reluctance to accept geometric algebra was the prevailing view amongst physicists that there is something intrinsically 'quantum mechanical' in the algebra. This is quite wrong, as witnessed by the fact that Clifford predated quantum theory by 50 years, but it has taken a long time for this to be widely appreciated.

In Cambridge today, we routinely apply geometric algebra to topics as diverse as black holes and cosmology, quantum tunnelling and quantum field theory, beam dynamics and buckling, robotics and computer vision. Exactly the same algebraic system is used throughout, making it possible for the same people to understand and contribute to all of these different fields.

1.8 Geometric Algebra in 2-d

The easiest way to understand the geometric product is by example, so consider a 2-d space (a plane) spanned by 2 orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2$. These basis vectors satisfy

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1, \quad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0. \quad (1.29)$$

We have now reverted to writing vectors in a bold face. We will use this convention from now on when working specifically in 2-d or 3-d. When working in larger spaces, or arbitrary dimensions, it is generally more convenient to drop this convention (otherwise one can end up with almost every term in a long expression written in bold face).

The final entity present in the 2-d algebra is the bivector $\mathbf{e}_1 \wedge \mathbf{e}_2$. This is the highest grade element in the algebra, which is often called the *pseudoscalar*. This name is more convenient than the technically more correct *directed volume element*. The pseudoscalar is by convention chosen to be *right-handed*, so that \mathbf{e}_1 sweeps onto \mathbf{e}_2 in a right-handed sense (when viewed from above). We use the symbol I for the pseudoscalar of a geometric algebra so in 2-d we write

$$I = \mathbf{e}_1 \wedge \mathbf{e}_2. \quad (1.30)$$

The full algebra is spanned by

$$\begin{array}{ccc} 1 & \{\mathbf{e}_1, \mathbf{e}_2\} & \mathbf{e}_1 \wedge \mathbf{e}_2 \\ 1 \text{ scalar} & 2 \text{ vectors} & 1 \text{ bivector.} \end{array} \quad (1.31)$$

We denote this algebra by \mathcal{G}_2 . To study the properties of the bivector $\mathbf{e}_1 \wedge \mathbf{e}_2$ we first note that

$$\mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2. \quad (1.32)$$

That is, for orthogonal vectors the geometric product is a pure bivector. Also note that

$$\mathbf{e}_2 \mathbf{e}_1 = \mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_1 \wedge \mathbf{e}_2 \quad (1.33)$$

from the antisymmetry of the exterior product. Another way of saying this is that *orthogonal vectors anticommute*.

We can now form products when $\mathbf{e}_1 \mathbf{e}_2 = I$ multiplies vectors from the left and the right. First from the left,

$$\begin{aligned} I \mathbf{e}_1 &= (-\mathbf{e}_2 \mathbf{e}_1) \mathbf{e}_1 = -\mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_1 = -\mathbf{e}_2 \\ I \mathbf{e}_2 &= (\mathbf{e}_1 \mathbf{e}_2) \mathbf{e}_2 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2 = \mathbf{e}_1. \end{aligned} \quad (1.34)$$

We see that left multiplication by the bivector rotates vectors 90° clockwise (*i.e.* in a negative sense). Similarly, acting from the right

$$\mathbf{e}_1 (\mathbf{e}_1 \mathbf{e}_2) = \mathbf{e}_2 \quad \mathbf{e}_2 (\mathbf{e}_1 \mathbf{e}_2) = -\mathbf{e}_1. \quad (1.35)$$

So right multiplication by I rotates 90° anticlockwise — a positive sense.

The final product in the algebra to consider is the square of the bivector $\mathbf{e}_1 \wedge \mathbf{e}_2$,

$$I^2 = (\mathbf{e}_1 \wedge \mathbf{e}_2)^2 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 = -\mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2 = -1. \quad (1.36)$$

From purely geometric considerations, we have discovered a quantity which squares to -1 . This fits with the fact that 2 successive left (or right) multiplications of a vector by $\mathbf{e}_1 \mathbf{e}_2$ rotates the vector through 180° , which is equivalent to multiplying by -1 .

1.8.1 Multiplying Multivectors

Suppose that we have two completely arbitrary elements of the \mathcal{G}_2 algebra, A and B . We can decompose these in terms of our $\{\mathbf{e}_1, \mathbf{e}_2\}$ frame as follows:

$$\begin{aligned} A &= a_0 + a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_1 \wedge \mathbf{e}_2 \\ B &= b_0 + b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_1 \wedge \mathbf{e}_2. \end{aligned} \quad (1.37)$$

The product of these two elements can be written

$$AB = p_0 + p_1 \mathbf{e}_1 + p_2 \mathbf{e}_2 + p_3 \mathbf{e}_1 \wedge \mathbf{e}_2. \quad (1.38)$$

We find that

$$\begin{aligned} p_0 &= a_0 b_0 + a_1 b_1 + a_2 b_2 - a_3 b_3 \\ p_1 &= a_0 b_1 + a_1 b_0 + a_3 b_2 - a_2 b_3 \\ p_2 &= a_0 b_2 + a_2 b_0 + a_1 b_3 - a_3 b_1 \\ p_3 &= a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1. \end{aligned} \quad (1.39)$$

This multiplication law is easy to represent as part of a computer language (we often use Maple). The basis vectors can also be represented with matrices, though these can hide the geometry of the algebra. If we introduce the symbol $\langle AB \rangle$ to denote the scalar term in the product, we see that

$$p_0 = \langle AB \rangle = \langle BA \rangle. \quad (1.40)$$

In general, however, $AB \neq BA$.

1.8.2 Complex Numbers and \mathcal{G}_2

It is clear that there is a close relationship between GA in 2-d, and the algebra of complex numbers. The unit bivector squares to -1 and generates rotations through 90° . The combination of a scalar and a bivector, which is formed naturally via the geometric product, can therefore be viewed as a complex number. We can write

$$z = x + y \mathbf{e}_1 \mathbf{e}_2 = x + Iy. \quad (1.41)$$

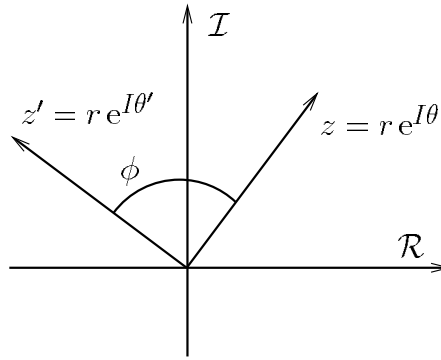


Figure 1.9: *A Rotation in the Complex Plane.* The overall effect is to replace θ by $\theta' = \theta + \phi$

Complex numbers serve a dual purpose in 2-d. They generate rotations and dilations through their polar decomposition $r \exp(i\theta)$, and they also represent vectors as points on the Argand diagram (Fig. 1.2). But in \mathcal{G}_2 our vectors are grade-1 objects,

$$\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2. \quad (1.42)$$

Is there a natural map between this and the complex number z ? The answer is simple — pre-multiply by \mathbf{e}_1 ,

$$\mathbf{e}_1\mathbf{x} = x + y\mathbf{e}_1\mathbf{e}_2 = x + Iy = z. \quad (1.43)$$

That is all there is to it! The role of the preferred vector \mathbf{e}_1 is clear — it is the real axis. Using this product vectors can be interchanged with complex numbers in 2-d in a natural manner.

1.8.3 Rotations

Since we know how to rotate complex numbers, we can use this to find a formula for rotating vectors in 2-d. We know that a positive rotation through an angle ϕ for a complex number z is achieved by

$$z \mapsto e^{I\phi} z, \quad (1.44)$$

where we continue to use I for the imaginary (see Fig. 1.9). The exponential of a multivector is defined by power series in the normal way. We can now apply this to the vector transformation $\mathbf{x} \mapsto \mathbf{x}'$ as follows

$$\begin{aligned} \mathbf{x} &= \mathbf{e}_1 z \mapsto \mathbf{x}' = \mathbf{e}_1 z' \\ \mathbf{x}' &= \mathbf{e}_1 e^{I\phi} z = e^{-I\phi} \mathbf{e}_1 z = e^{-I\phi} \mathbf{x}. \end{aligned} \quad (1.45)$$

We therefore arrive at the formulae

$$\boldsymbol{x}' = e^{-I\phi} \boldsymbol{x} = \boldsymbol{x} e^{I\phi}. \quad (1.46)$$

As expected, multiplication from the right by I generates a positive rotation in the I plane. Note the importance of the fact that I *anticommutes* with vectors. We do not get behaviour like this with complex numbers alone.

The GA treatment shows us how complex numbers are able to play two roles, as rotations/dilations, and as position vectors. GA separates these roles, which is crucial to understanding how to generalise complex algebra to higher dimensions. In particular, these considerations generalise to a new and powerful way to handle arbitrary rotations.

Chapter 2

Geometric Algebra of 3-d Space

The geometric algebra (GA) of 3-d space is a remarkably powerful tool for solving problems in geometry and classical mechanics. It describes vectors, planes and volumes in a single algebra, which contains all of the familiar vector operations for 3-d space. These include the vector cross product, which is revealed as a disguised form of bivector. The algebra provides a very clear and compact method for encoding rotations, which is considerably more powerful than working with matrices. This reveals the true significance of Hamilton's quaternions, and resolves many of the historical difficulties encountered with their use.

2.1 Geometric Algebra in 3-d

In Chapter 1 we constructed the geometric algebra of a 2-d plane. We now add a third vector \mathbf{e}_3 to our 2-d set $\{\mathbf{e}_1, \mathbf{e}_2\}$. All three vectors are assumed to be orthonormal, so they all *anticommute* and square to $+1$. From these 3 basis vectors we can generate 3 independent bivectors:

$$\mathbf{e}_1\mathbf{e}_2, \quad \mathbf{e}_2\mathbf{e}_3, \quad \text{and} \quad \mathbf{e}_3\mathbf{e}_1. \quad (2.1)$$

Each of these encodes a distinct plane, and there are 3 of them to match the 3 independent planes in 3-d space.

As well as the 3 bivectors our expanded algebra gives us one new product to consider. This is the product of 3 orthogonal vectors, resulting in

$$(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3. \quad (2.2)$$

This corresponds to sweeping the bivector $\mathbf{e}_1\mathbf{e}_2$ along the vector \mathbf{e}_3 . The result is a 3-dimensional volume element and is called a *trivector*. This is said to have *grade-3*,

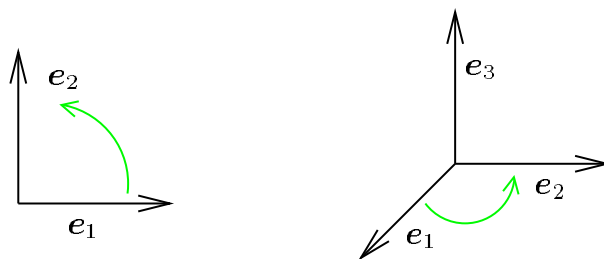


Figure 2.1: *Handedness*. The two frames shown are, by convention, assigned a right-handed orientation. Both $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ give rise to right-handed pseudoscalars for their respective algebras.

where the word ‘grade’ refers to the number of independent vectors forming the object. So vectors are grade-1, bivectors are grade-2, and so on. The term ‘grade’ is preferred to ‘dimension’ as the latter is reserved for the size of a linear space.

In 3-d the maximum number of independent vectors is 3, so the trivector is the highest grade object, or *multivector*, in the algebra. This trivector is unique up to scale (*i.e.* volume) and handedness (see below). As in the 2-d case the unit highest-grade multivector is called the *pseudoscalar*, or *directed volume element*. The latter name is more accurate, but the former is seen more often. (Though be careful with this usage — pseudoscalar can mean different things in different contexts). To simplify, we introduce the symbol I ,

$$I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3. \quad (2.3)$$

Our 3-d algebra is therefore spanned by

$$\begin{array}{cccc} 1 & \{\mathbf{e}_i\} & \{\mathbf{e}_i \wedge \mathbf{e}_j\} & I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\ 1 \text{ scalar} & 3 \text{ vectors} & 3 \text{ bivectors} & 1 \text{ trivector} \end{array} \quad (2.4)$$

These define a linear space of dimension $8 = 2^3$. We call this algebra \mathcal{G}_3 . Notice that the dimensions of each subspace are given by the binomial coefficients.

As in 2-d, the pseudoscalar is, by convention, chosen to be *right-handed*. This is equivalent to saying that the generating frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is right-handed. If a left-handed set of orthonormal vectors is multiplied together the result is $-I$. There is no intrinsic definition of handedness — it is a convention adopted to make our life easier for 2-d and 3-d. In 2-d the set $\{\mathbf{e}_1, \mathbf{e}_2\}$ are right-handed if, viewed from above, \mathbf{e}_1 rotates onto \mathbf{e}_2 in a right-handed sense. Note that this definition involves a notion of above and below, so is not intrinsic to the surface. In 3-d the usual right-handed rule is adopted. Align the thumb along the \mathbf{e}_3 direction. Then the grip of your right hand specifies the direction in which \mathbf{e}_1 rotates onto \mathbf{e}_2 (Fig. 2.1). The handedness of a frame changes sign if the positions of any two vectors are swapped.

2.2 Products in \mathcal{G}_3

Any two vectors in the algebra, \mathbf{a} and \mathbf{b} say, can be multiplied with the geometric product, and we still have

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}. \quad (2.5)$$

Now the bivector $\mathbf{a} \wedge \mathbf{b}$ belongs to a 3-d space, spanned by the $\{\mathbf{e}_i \wedge \mathbf{e}_j\}$. If we expand out in a basis,

$$\mathbf{a} = \sum_{i=1}^3 a_i \mathbf{e}_i, \quad \mathbf{b} = \sum_{i=1}^3 b_i \mathbf{e}_i, \quad (2.6)$$

we find that the components of the outer product are given by

$$\mathbf{a} \wedge \mathbf{b} = (a_2 b_3 - a_3 b_2) \mathbf{e}_2 \wedge \mathbf{e}_3 + (a_3 b_1 - a_1 b_3) \mathbf{e}_3 \wedge \mathbf{e}_1 + (a_1 b_2 - a_2 b_1) \mathbf{e}_1 \wedge \mathbf{e}_2. \quad (2.7)$$

The components are the same as those of the cross product, but the result is a bivector rather than a vector. To understand the relationship between these we first need to establish the properties of some of the new products provided by our 3-d algebra.

2.2.1 Vectors and Bivectors

The three basis bivectors share the properties of the 2-d bivector studied in Chapter 1. In particular,

$$(\mathbf{e}_1 \mathbf{e}_2)^2 = (\mathbf{e}_2 \mathbf{e}_3)^2 = (\mathbf{e}_3 \mathbf{e}_1)^2 = -1 \quad (2.8)$$

and each bivector generates 90° rotations in its own plane. So, for example, we recall that

$$\mathbf{e}_1(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_1(\mathbf{e}_1 \mathbf{e}_2) = \mathbf{e}_2, \quad (2.9)$$

which returns a vector. The geometric product for vectors extends to all objects in the algebra, so we can form expressions such as $\mathbf{a}B$, where B is a general bivector. But we have now seen that $\mathbf{e}_1(\mathbf{e}_2 \wedge \mathbf{e}_3)$ is a trivector, so the result of the product $\mathbf{a}B$ can clearly contain both vector and trivector terms. To help understand the properties of the product $\mathbf{a}B$ we first decompose \mathbf{a} into terms in and out of the plane,

$$\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp} \quad (2.10)$$

(see Fig. 2.2). We can now write $\mathbf{a}B = (\mathbf{a}_{\parallel} + \mathbf{a}_{\perp})B$. Suppose that we also write

$$B = \mathbf{a}_{\parallel} \wedge \mathbf{b} \quad (2.11)$$

where \mathbf{b} is orthogonal to \mathbf{a}_{\parallel} in the B plane (Fig. 2.2). We see that

$$\mathbf{a}_{\parallel}B = \mathbf{a}_{\parallel}(\mathbf{a}_{\parallel} \wedge \mathbf{b}) = \mathbf{a}_{\parallel}(\mathbf{a}_{\parallel} \mathbf{b}) = (\mathbf{a}_{\parallel})^2 \mathbf{b} \quad (2.12)$$

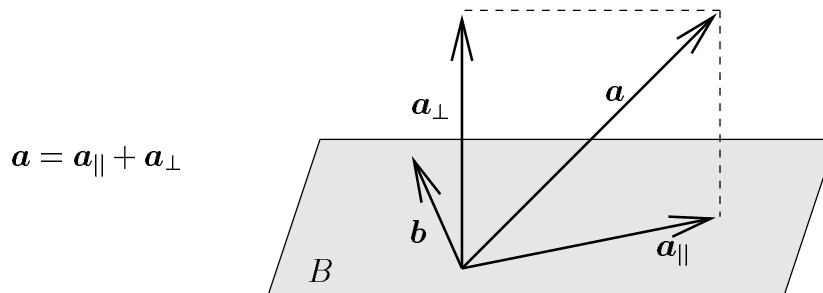


Figure 2.2: *A vector and a plane.* The vector \mathbf{a} is decomposed into a sum of two vectors, one lying in the plane and the other perpendicular to it.

which is a vector in the \mathbf{b} direction. On the other hand

$$\mathbf{a}_{\perp}B = \mathbf{a}_{\perp}(\mathbf{a}_{\parallel} \wedge \mathbf{b}) = \mathbf{a}_{\perp} \mathbf{a}_{\parallel} \mathbf{b} \quad (2.13)$$

is the geometric product of 3 orthogonal vectors, and so is a trivector. As expected, the geometric product of the vector \mathbf{a} and the bivector B has resulted in two terms, a vector and a trivector. We therefore write

$$\mathbf{a}B = \mathbf{a} \cdot B + \mathbf{a} \wedge B \quad (2.14)$$

where the dot is generalised to mean the *lowest* grade part of the result, while the wedge means the *highest* grade part of the result.

2.2.2 Inner Product $\mathbf{a} \cdot B$

From Eq. (2.12) we see that the $\mathbf{a} \cdot B = \mathbf{a}_{\parallel} \cdot B$ term projects onto the component of \mathbf{a} in the plane, and then rotates this through 90° and dilates by the magnitude of B . We also see that

$$\mathbf{a} \cdot B = \mathbf{a}_{\parallel}^2 \mathbf{b} = -(\mathbf{a}_{\parallel} \mathbf{b}) \mathbf{a}_{\parallel} = -B \cdot \mathbf{a}, \quad (2.15)$$

so the dot product between a vector and a bivector is antisymmetric. We use this to *define* the inner product of a vector and a bivector as

$$\mathbf{a} \cdot B = \frac{1}{2}(\mathbf{a}B - B\mathbf{a}). \quad (2.16)$$

To see that this always returns a vector, consider the inner product $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$. Following the rules for the geometric product we form:

$$\begin{aligned} \mathbf{a}(\mathbf{b} \wedge \mathbf{c}) &= \frac{1}{2} \mathbf{a}(\mathbf{bc} - \mathbf{cb}) \\ &= (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - \frac{1}{2}(\mathbf{bac} - \mathbf{cab}) \\ &= 2(\mathbf{a} \cdot \mathbf{b})\mathbf{c} - 2(\mathbf{a} \cdot \mathbf{c})\mathbf{b} + \frac{1}{2}(\mathbf{bc} - \mathbf{cb})\mathbf{a} \\ &= 2(\mathbf{a} \cdot \mathbf{b})\mathbf{c} - 2(\mathbf{a} \cdot \mathbf{c})\mathbf{b} + (\mathbf{b} \wedge \mathbf{c})\mathbf{a}, \end{aligned} \quad (2.17)$$

where we have made repeated use of the rearrangement

$$\mathbf{ba} = 2\mathbf{a}\cdot\mathbf{b} - \mathbf{ab}. \quad (2.18)$$

It follows immediately that

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) = \frac{1}{2}(\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) - (\mathbf{b}\wedge\mathbf{c})\mathbf{a}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b}, \quad (2.19)$$

which is indeed a pure vector. This is one of the most useful results in geometric algebra and is worth memorising.

2.2.3 Outer Product $\mathbf{a}\wedge\mathbf{B}$

From Eq. (2.13), the $\mathbf{a}\wedge\mathbf{B}$ term projects onto the component perpendicular to the plane, and returns a trivector. This term is symmetric

$$\mathbf{a}\wedge\mathbf{B} = \mathbf{a}_\perp\mathbf{a}_\parallel\mathbf{b} = \mathbf{a}_\parallel\mathbf{b}\mathbf{a}_\perp = \mathbf{B}\wedge\mathbf{a}. \quad (2.20)$$

We therefore define the outer product of a vector and a bivector as

$$\mathbf{a}\wedge\mathbf{B} = \frac{1}{2}(\mathbf{a}\mathbf{B} + \mathbf{B}\mathbf{a}). \quad (2.21)$$

Various arguments can be used to show that this is a pure trivector (see later). We now have a definition of the outer product of three vectors, $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$. This is the grade-3 part of the geometric product. We denote the operation of projecting onto the terms of a given grade with the $\langle \rangle_r$ symbol, where r is the required grade. Using this we can write

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \langle \mathbf{a}(\mathbf{b}\wedge\mathbf{c}) \rangle_3 = \langle \mathbf{a}(\mathbf{bc} - \mathbf{b}\cdot\mathbf{c}) \rangle_3. \quad (2.22)$$

But in the final term $\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$ is a vector (grade-1) so does not contribute. It follows that

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \langle \mathbf{a}(\mathbf{bc}) \rangle_3 = \langle \mathbf{abc} \rangle_3, \quad (2.23)$$

where we have used the fact that the geometric product is associative to remove the brackets. It follows from this simple derivation that the outer product is also associative,

$$(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} = \mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}. \quad (2.24)$$

This is true in general.

The trivector $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$ can be pictured as the parallelepiped formed by sweeping $\mathbf{a}\wedge\mathbf{b}$ along \mathbf{c} (see Fig. 2.3). The same result is obtained by sweeping $\mathbf{b}\wedge\mathbf{c}$ along \mathbf{a} , which is the geometric way of picturing the associativity of the outer product. The other main property of the outer product is that it is antisymmetric on every pair of vectors,

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = -\mathbf{b}\wedge\mathbf{a}\wedge\mathbf{c} = \mathbf{c}\wedge\mathbf{a}\wedge\mathbf{b}, \quad \text{etc.} \quad (2.25)$$

This expresses the geometric result that swapping any two vectors reverses the orientation (handedness) of the product.

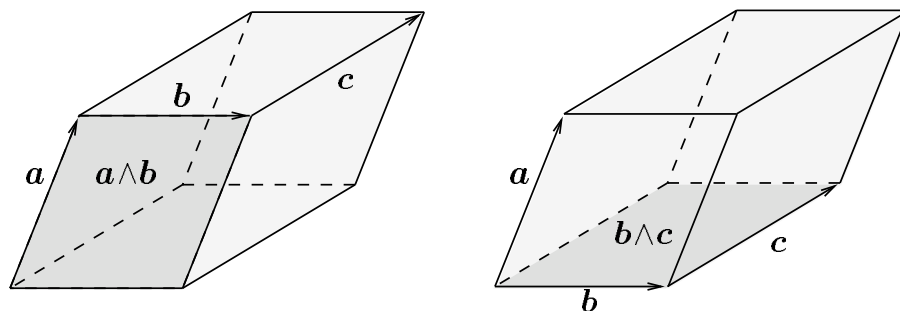


Figure 2.3: *The Trivector*. The result of sweeping $\mathbf{a} \wedge \mathbf{b}$ along \mathbf{c} is a directed volume, or trivector. The same trivector is obtained by sweeping $\mathbf{b} \wedge \mathbf{c}$ along \mathbf{a} .

2.2.4 The Bivector Algebra

Our three independent bivectors also give us a further new product to consider. When multiplying two bivectors we find, for example, that

$$(\mathbf{e}_1 \wedge \mathbf{e}_2)(\mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_1 \mathbf{e}_3, \quad (2.26)$$

resulting in a third bivector. We also find that

$$(\mathbf{e}_2 \wedge \mathbf{e}_3)(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3 \mathbf{e}_2 \mathbf{e}_2 \mathbf{e}_1 = \mathbf{e}_3 \mathbf{e}_1 = -\mathbf{e}_1 \mathbf{e}_3, \quad (2.27)$$

so the product is antisymmetric. The symmetric contribution vanishes because the two planes are perpendicular. If we introduce the following labelling for the basis bivectors:

$$B_1 = \mathbf{e}_2 \mathbf{e}_3, \quad B_2 = \mathbf{e}_3 \mathbf{e}_1, \quad B_3 = \mathbf{e}_1 \mathbf{e}_2 \quad (2.28)$$

we find that the commutator satisfies

$$B_i B_j - B_j B_i = -2\epsilon_{ijk} B_k. \quad (2.29)$$

This algebra is closely linked to 3-d rotations, and will be familiar from the quantum theory of angular momentum.

It is useful to introduce a symbol for one-half the commutator of 2 bivectors. We call this the *commutator product* and denote it with a cross, so

$$A \times B = \frac{1}{2}(AB - BA). \quad (2.30)$$

The commutator product of two bivectors always results in a third bivector (or zero). We will learn more about this later.

The basis bivectors all square to -1 , and all anticommute. These are the properties of the generators of the quaternion algebra (see Section 1.4). This observation helps to

sort out some of the problems encountered with the quaternions. Hamilton attempted to identify pure quaternions (null scalar part) with vectors, but we now see that they are actually *bivectors*. This has an important consequence when we look at their behaviour under reflections. Hamilton also imposed the condition $\mathbf{i}\mathbf{j}\mathbf{k} = -1$ on his unit quaternions, whereas we have

$$B_1 B_2 B_3 = \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_3 \mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_2 = +1. \quad (2.31)$$

To set up a direct map we must flip a sign somewhere, for example in the y component:

$$\mathbf{i} \leftrightarrow B_1, \quad \mathbf{j} \leftrightarrow -B_2, \quad \mathbf{k} \leftrightarrow B_3. \quad (2.32)$$

This shows us that the quaternions were *left-handed*, even though the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ were interpreted as a right-handed set of vectors. Not surprisingly, this was a source of some confusion!

2.2.5 Products Involving the Pseudoscalar

The pseudoscalar $I = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$ is the unique right-handed unit trivector in the algebra. This gives us a number of new products to consider. We start by forming the product of I with the vector \mathbf{e}_1 ,

$$I \mathbf{e}_1 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_1 = -\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_3 = \mathbf{e}_2 \mathbf{e}_3. \quad (2.33)$$

The result is a bivector — the plane perpendicular to the original vector (see Fig. 2.4). The product of a grade-1 vector with the grade-3 pseudoscalar is therefore a grade-2 bivector. Reversing the order we find that

$$\mathbf{e}_1 I = \mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_2 \mathbf{e}_3. \quad (2.34)$$

The result is therefore independent of order — the pseudoscalar commutes with all vectors in 3-d,

$$I \mathbf{a} = \mathbf{a} I, \quad \text{for all } \mathbf{a}. \quad (2.35)$$

It follows that I commutes with all elements in the algebra. This is always the case for the pseudoscalar in spaces of odd dimension. In even dimensions, the pseudoscalar anticommutes with all vectors, as we have already seen in 2-d. We can now express each of our basis bivectors as the product of the pseudoscalar and a *dual* vector,

$$\mathbf{e}_1 \mathbf{e}_2 = I \mathbf{e}_3, \quad \mathbf{e}_2 \mathbf{e}_3 = I \mathbf{e}_1, \quad \mathbf{e}_3 \mathbf{e}_1 = I \mathbf{e}_2. \quad (2.36)$$

This operation of multiplying by the pseudoscalar is called a *duality* transformation.

We next form the square of the pseudoscalar

$$I^2 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 = -1. \quad (2.37)$$

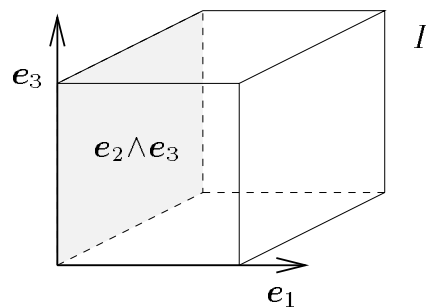


Figure 2.4: *The product of a vector and a trivector.* The diagram shows the result of the product $\mathbf{e}_1 I = \mathbf{e}_1(\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3) = \mathbf{e}_2 \mathbf{e}_3$

So the pseudoscalar commutes with all elements and squares to -1 . It is therefore a further candidate for a unit imaginary. In some physical applications this is the correct one to use, whereas for others it is one of the bivectors. These different possibilities provide us with a very rich geometric language.

Finally, we consider the product of a bivector and the pseudoscalar:

$$I(\mathbf{e}_1 \wedge \mathbf{e}_2) = I\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_3 = I\mathbf{e}_3 = -\mathbf{e}_3. \quad (2.38)$$

So the result of the product of I with the bivector formed from \mathbf{e}_1 and \mathbf{e}_2 is $-\mathbf{e}_3$, that is, minus the vector perpendicular to the $\mathbf{e}_1 \wedge \mathbf{e}_2$ plane. This affords a definition of the vector cross product in 3-d as

$$\mathbf{a} \times \mathbf{b} = -I(\mathbf{a} \wedge \mathbf{b}). \quad (2.39)$$

The bold \times symbol should not be confused with the \times symbol for the commutator product. The latter is extremely useful, whereas the vector cross product is largely redundant now that we have the outer product available. Equation (2.39) shows how the cross product is a bivector in disguise, the bivector being mapped to a vector by a duality operation. It is also now clear why the product only exists in 3-d — this is the only space for which the dual of a bivector is a vector. We will have little further use for the cross product and will rarely employ it from now on. This means we can also do away with the awkward distinction between axial and polar vectors. Instead we just talk of vectors and bivectors.

The duality operation in 3-d provides an alternative way to understand the geometric product $\mathbf{a}B$ of a vector and a bivector. We write $B = I\mathbf{b}$ in terms of its dual vector \mathbf{b} , so that we now have

$$\mathbf{a}B = I\mathbf{a}\mathbf{b} = I(\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}). \quad (2.40)$$

This demonstrates that the symmetric part of the product generates the trivector

$$\mathbf{a} \wedge B = I(\mathbf{a} \cdot \mathbf{b}) = \frac{1}{2}(\mathbf{a}B + B\mathbf{a}), \quad (2.41)$$

whereas the antisymmetric part returns a vector

$$\mathbf{a} \cdot B = I(\mathbf{a} \wedge \mathbf{b}) = \frac{1}{2}(\mathbf{a}B - B\mathbf{a}). \quad (2.42)$$

This justifies the definition of the inner and outer products between a vector and bivector. As with pairs of vectors, these combine to return the geometric product,

$$\mathbf{a}B = \mathbf{a} \cdot B + \mathbf{a} \wedge B. \quad (2.43)$$

2.3 Further Definitions

An important operation in GA is that of reversing the order of vectors in any product. This is denoted with a tilde, $\tilde{\cdot}$. Scalars and vectors are invariant under reversion, but bivectors change sign,

$$(\mathbf{e}_1\mathbf{e}_2)^\sim = \mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2. \quad (2.44)$$

Similarly, we see that

$$\tilde{I} = \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_3\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -I. \quad (2.45)$$

A general multivector in 3-d can be written

$$M = \alpha + \mathbf{a} + B + \beta I. \quad (2.46)$$

From the above we see that

$$\tilde{M} = \alpha + \mathbf{a} - B - \beta I. \quad (2.47)$$

It is also useful to adopt the *operator ordering convention* that, in the absence of brackets, *inner and outer products are performed before geometric products*. This cleans up expressions by enabling us to remove unnecessary brackets. For example, on the right-hand side of Eq. (2.39) we can now write

$$\mathbf{a} \times \mathbf{b} = -I \mathbf{a} \wedge \mathbf{b}. \quad (2.48)$$

We have already introduced the $\langle \rangle_r$ notation for projecting onto the terms of grade- r . For the operation of projecting onto the scalar component we usually drop the subscript 0 and write

$$\langle AB \rangle = \langle AB \rangle_0 \quad (2.49)$$

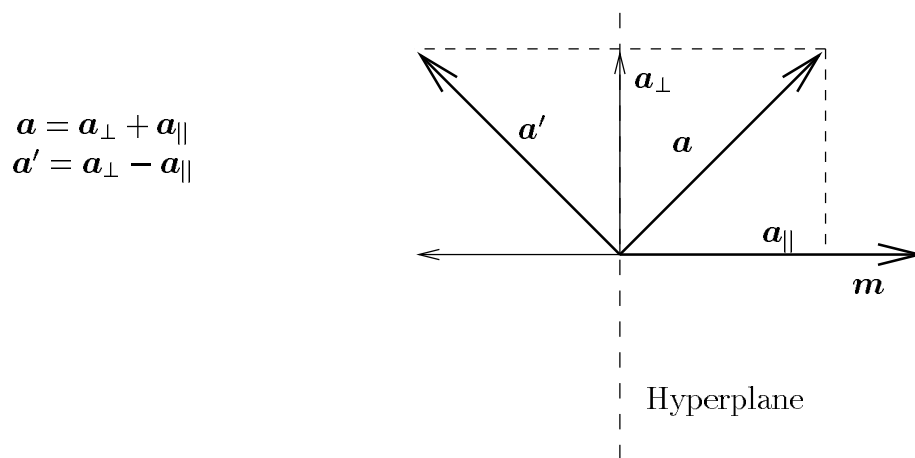
for the scalar part of the product of two arbitrary multivectors. The scalar product is always symmetric

$$\langle AB \rangle = \langle BA \rangle. \quad (2.50)$$

It follows that

$$\langle A \cdots BC \rangle = \langle CA \cdots B \rangle. \quad (2.51)$$

This cyclic reordering property is very useful in practice.

Figure 2.5: A reflection in the plane perpendicular to \mathbf{m} .

2.4 Reflections

Suppose that we reflect the vector \mathbf{a} in the (hyper)plane orthogonal to some unit vector \mathbf{m} ($\mathbf{m}^2 = 1$). The component of \mathbf{a} parallel to \mathbf{m} changes sign, whereas the perpendicular component is unchanged. The parallel component is the projection onto \mathbf{m} :

$$\mathbf{a}_\parallel = \mathbf{a} \cdot \mathbf{m} \mathbf{m}. \quad (2.52)$$

(NB operator ordering convention in force here.) The perpendicular component is the remainder

$$\mathbf{a}_\perp = \mathbf{a} - \mathbf{a} \cdot \mathbf{m} \mathbf{m} = (\mathbf{a} \mathbf{m} - \mathbf{a} \cdot \mathbf{m}) \mathbf{m} = \mathbf{a} \wedge \mathbf{m} \mathbf{m}. \quad (2.53)$$

This shows how the wedge product projects onto the components perpendicular to a vector. The result of the reflection is therefore

$$\begin{aligned} \mathbf{a}' &= \mathbf{a}_\perp - \mathbf{a}_\parallel = -\mathbf{a} \cdot \mathbf{m} \mathbf{m} + \mathbf{a} \wedge \mathbf{m} \mathbf{m} \\ &= -(\mathbf{m} \cdot \mathbf{a} + \mathbf{m} \wedge \mathbf{a}) \mathbf{m} = -\mathbf{m} \mathbf{a} \mathbf{m}. \end{aligned} \quad (2.54)$$

This remarkably compact formula only arises in geometric algebra. We can start to see now that geometric products arise naturally when *operating* on vectors.

It is simple to check that our formula has the required properties. For any vector $\lambda \mathbf{m}$ in the \mathbf{m} direction we have

$$-\mathbf{m}(\lambda \mathbf{m})\mathbf{m} = -\lambda \mathbf{m} \mathbf{m} \mathbf{m} = -\lambda \mathbf{m} \quad (2.55)$$

and so $\lambda \mathbf{m}$ is reflected. Similarly, for any vector \mathbf{n} perpendicular to \mathbf{m} we have

$$-\mathbf{m}(\mathbf{n})\mathbf{m} = -\mathbf{m} \mathbf{n} \mathbf{m} = \mathbf{n} \mathbf{m} \mathbf{m} = \mathbf{n} \quad (2.56)$$

and so \mathbf{n} is unaffected. We can also give a simple proof that inner products are unchanged by reflections,

$$\begin{aligned}\mathbf{a}' \cdot \mathbf{b}' &= (-\mathbf{mam}) \cdot (-\mathbf{mbm}) = \langle \mathbf{mammbm} \rangle \\ &= \langle \mathbf{mabm} \rangle = \langle \mathbf{mmab} \rangle = \mathbf{a} \cdot \mathbf{b}.\end{aligned}\tag{2.57}$$

We next construct the transformation law for the bivector $\mathbf{a} \wedge \mathbf{b}$ under reflection of both \mathbf{a} and \mathbf{b} . We obtain

$$\begin{aligned}\mathbf{a}' \wedge \mathbf{b}' &= (-\mathbf{mam}) \wedge (-\mathbf{mbm}) = \frac{1}{2}(\mathbf{mammbm} - \mathbf{mbmmam}) \\ &= \frac{1}{2}\mathbf{m}(\mathbf{ab} - \mathbf{ba})\mathbf{m} = \mathbf{m} \mathbf{a} \wedge \mathbf{b} \mathbf{m}.\end{aligned}\tag{2.58}$$

We recover essentially the same law, but with a crucial sign difference. Bivectors do not transform as vectors under reflections. This is the reason for the confusing distinction between polar and axial vectors in 3-d. Axial vectors invariably arise as the result of the cross product. They are really bivectors and should be treated as such. This also explains why 19th century mathematicians were confused by the transformation properties of the quaternions. They were expected to transform as vectors under reflections, but actually transform as bivectors (*i.e.* with the opposite sign).

2.5 Rotations

In Section 1.8.3 we saw that in 2-d a vector can be rotated through θ in the $\mathbf{e}_1\mathbf{e}_2$ plane by

$$\mathbf{a} \mapsto \mathbf{a}' = \exp(-\mathbf{e}_1\mathbf{e}_2\theta)\mathbf{a} = \mathbf{a} \exp(\mathbf{e}_1\mathbf{e}_2\theta).\tag{2.59}$$

We now want to find a version of this formula appropriate for 3-d, a problem with which Hamilton struggled for many years. The key to finding the correct formula is to use that result that *a rotation in the plane generated by two unit vectors \mathbf{m} and \mathbf{n} is achieved by successive reflections in the (hyper)planes perpendicular to \mathbf{m} and \mathbf{n} .* This is illustrated in Fig. 2.6. It is clear that any component of \mathbf{a} outside the $\mathbf{m} \wedge \mathbf{n}$ plane is untouched. It is also a simple exercise in trigonometry to confirm that the angle between the initial vector \mathbf{a} and the final vector \mathbf{a}'' is twice the angle between \mathbf{m} and \mathbf{n} . (This is left as an exercise.) The result of the successive reflections is therefore to rotate through 2θ in the $\mathbf{m} \wedge \mathbf{n}$ plane, where $\mathbf{m} \cdot \mathbf{n} = \cos(\theta)$.

So how does this look in GA?

$$\mathbf{a}' = -\mathbf{mam}\tag{2.60}$$

$$\mathbf{a}'' = -\mathbf{na}'\mathbf{n} = -\mathbf{n}(-\mathbf{mam})\mathbf{n} = \mathbf{nmam}\mathbf{n}\tag{2.61}$$

This is beginning to look very simple! We define the *rotor* R by

$$R = \mathbf{nm}.\tag{2.62}$$

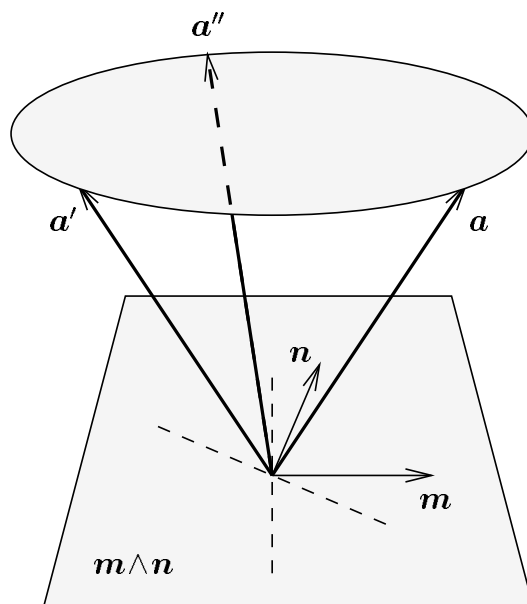


Figure 2.6: *A Rotation from 2 Reflections.* \mathbf{a}' is the result of reflecting \mathbf{a} in the plane perpendicular to \mathbf{m} . \mathbf{a}'' is the result of reflecting \mathbf{a}' in the plane perpendicular to \mathbf{n} .

Note the *geometric* product here! We can now write a rotation as

$$\mathbf{a} \mapsto R\mathbf{a}\tilde{R} \quad (2.63)$$

Incredibly, this formula works for any grade of multivector, in any dimension, of any signature! To make contact with the 2-d result we first expand R as

$$R = \mathbf{nm} = \mathbf{n} \cdot \mathbf{m} + \mathbf{n} \wedge \mathbf{m} = \cos(\theta) + \mathbf{n} \wedge \mathbf{m}. \quad (2.64)$$

So what is the magnitude of the bivector $\mathbf{n} \wedge \mathbf{m}$?

$$\begin{aligned} (\mathbf{n} \wedge \mathbf{m}) \cdot (\mathbf{n} \wedge \mathbf{m}) &= \langle \mathbf{n} \wedge \mathbf{m} \mathbf{n} \wedge \mathbf{m} \rangle \\ &= \langle \mathbf{nm} \mathbf{n} \wedge \mathbf{m} \rangle \\ &= \mathbf{n} \cdot [\mathbf{m} \cdot (\mathbf{n} \wedge \mathbf{m})] \\ &= \mathbf{n} \cdot (\mathbf{m} \cos(\theta) - \mathbf{n}) \\ &= \cos^2(\theta) - 1 = -\sin^2(\theta). \end{aligned} \quad (2.65)$$

We therefore define a unit bivector in the $\mathbf{m} \wedge \mathbf{n}$ plane by

$$\hat{B} = \mathbf{m} \wedge \mathbf{n} / \sin(\theta), \quad \hat{B}^2 = -1. \quad (2.66)$$

This choice of orientation ($\mathbf{m} \wedge \mathbf{n}$ rather than $\mathbf{n} \wedge \mathbf{m}$) ensures that the bivector has the same orientation as the rotation, as can be seen in Fig. 2.6.

In terms of the bivector \hat{B} we now have

$$R = \cos(\theta) - \hat{B} \sin(\theta). \quad (2.67)$$

Look familiar? This is nothing else than the polar decomposition of a complex number, with the unit imaginary replaced by the unit bivector \hat{B} . We can therefore write

$$R = \exp(-\hat{B}\theta). \quad (2.68)$$

The exponential here is defined in terms of its power series in the normal way. It is possible to show that this series is absolutely convergent for any multivector argument. (Exponentiating a multivector is essentially the same as exponentiating a matrix).

Now recall that our formula was for a rotation through 2θ . If we want to rotate through θ , the appropriate rotor is

$$R = \exp\{-\hat{B}\theta/2\} \quad (2.69)$$

which gives us the final formula

$$\mathbf{a} \mapsto e^{-\hat{B}\theta/2} \mathbf{a} e^{\hat{B}\theta/2}. \quad (2.70)$$

This describes a rotation through θ in the \hat{B} plane, with orientation specified by \hat{B} . The GA description forces us to think of rotations taking place *in a plane* as opposed to about an axis. The latter is an entirely 3-d concept, whereas the concept of a plane is quite general.

Rotors are one of the fundamental concepts in geometric algebra, and we will return to their properties many times throughout this course. Since the rotor R is a geometric product of two unit vectors, we see immediately that

$$R\tilde{R} = \mathbf{nm}(\mathbf{nm})^\sim = \mathbf{nm}m\mathbf{n} = 1 = \tilde{R}R. \quad (2.71)$$

This provides a quick proof that our formula has the correct property of preserving lengths and angles,

$$\mathbf{a}' \cdot \mathbf{b}' = (\mathbf{Ra}\tilde{R}) \cdot (\mathbf{Rb}\tilde{R}) = \langle \mathbf{Ra}\tilde{R}\mathbf{Rb}\tilde{R} \rangle = \langle \mathbf{Rab}\tilde{R} \rangle = \mathbf{a} \cdot \mathbf{b}. \quad (2.72)$$

Now suppose that the two vectors forming the bivector $B = \mathbf{a} \wedge \mathbf{b}$ are both rotated. What is the expression for the resulting bivector? To find this we form

$$\begin{aligned} B' &= \mathbf{a}' \wedge \mathbf{b}' = \frac{1}{2}(\mathbf{a}'\mathbf{b}' - \mathbf{b}'\mathbf{a}') = \frac{1}{2}(\mathbf{Ra}\tilde{R}\mathbf{Rb}\tilde{R} - \mathbf{Rb}\tilde{R}\mathbf{Ra}\tilde{R}) \\ &= \frac{1}{2}(\mathbf{Rab}\tilde{R} - \mathbf{Rba}\tilde{R}) = \frac{1}{2}R(\mathbf{ab} - \mathbf{ba})\tilde{R} = R\mathbf{a} \wedge \mathbf{b}\tilde{R} = RB\tilde{R}. \end{aligned} \quad (2.73)$$

Bivectors are rotated using precisely the same formula as vectors! The same turns out to be true for all geometric objects represented by multivectors. This is one of the most attractive features of geometric algebra.

Returning to the earlier 2-d result we see that we can write

$$\begin{aligned}\mathbf{a} \mapsto \mathbf{a}' &= \exp(-\mathbf{e}_1\mathbf{e}_2\theta)\mathbf{a} = \mathbf{a}\exp(\mathbf{e}_1\mathbf{e}_2\theta) \\ &= \exp(-\mathbf{e}_1\mathbf{e}_2\theta/2)\mathbf{a}\exp(\mathbf{e}_1\mathbf{e}_2\theta/2),\end{aligned}\tag{2.74}$$

which recovers the double-sided, half-angle formula. This has a surprising consequence. We are used to thinking of the unit imaginary as a generator for 90° rotations, which is the result of a single-sided application. Applied double-sidedly, however, the unit imaginary generates 180° rotations, by just changing sign. In 2-d we have a choice of how to work, but in higher dimensions it is only the double-sided formula that works. If we try to rotate \mathbf{e}_3 in the $\mathbf{e}_1\mathbf{e}_2$ plane with a single-sided formula the result is

$$\begin{aligned}e^{-\mathbf{e}_1\mathbf{e}_2\theta}\mathbf{e}_3 &= [\cos(\theta) - \sin(\theta)\mathbf{e}_1\mathbf{e}_2]\mathbf{e}_3 \\ &= \cos(\theta)\mathbf{e}_3 - \sin(\theta)I,\end{aligned}\tag{2.75}$$

which is wrong. In 3-d we therefore have to interpret the basis unit bivectors as generators of 180° rotations, in which case their algebraic relations are sensible. Hamilton and his followers wanted to maintain a single-sided transformation law and interpret the quaternions as generators of 90° rotations. This was a major mistake which held back the subject for many years.

Chapter 3

Classical Mechanics

We are now ready to see how geometric algebra (GA) is applied in practical physical problems. In this Chapter we concentrate on classical mechanics, looking at central force problems and rigid body mechanics. The purpose here is to highlight the areas where GA offers something new, either in understanding or in solution strategies. This is done here in a setting which allows you to see how GA techniques are applied to familiar problems. Later in this course we will move onto less familiar territory, and look at areas of quantum physics, relativity and gravitation.

3.1 Force and Angular Momentum

In classical mechanics a point particle follows a trajectory $\mathbf{x}(t)$, where \mathbf{x} is the position vector relative to some origin, and the time t is taken as some absolute ‘Newtonian’ standard on which all observers agree. The particle has velocity $\mathbf{v} = \dot{\mathbf{x}}$ where the overdot denotes differentiation with respect to time t . The momentum \mathbf{p} is given by $m\mathbf{v}$, and Newton’s second law states that $\dot{\mathbf{p}} = \mathbf{f}$, where the vector \mathbf{f} is the force acting on the particle. We assume here that the mass m is constant, so we recover $\mathbf{f} = m\mathbf{a}$, where \mathbf{a} is the acceleration $\mathbf{a} = \dot{\mathbf{v}}$.

These definitions should be (extremely) familiar. Our first departure from traditional methods comes with the treatment of angular momentum. This is usually defined in terms of the cross product,

$$\mathbf{l} = \mathbf{x} \times \mathbf{p}. \tag{3.1}$$

But it is clear that the concept of angular momentum should encode the rate at which area is swept out by a particle as it moves relative to some origin (see Figure 3.1). Areas are encoded by *bivectors*, as we saw in Chapter 2. We also saw there that the cross product is really a disguised form of bivector, mapped back to a vector by a

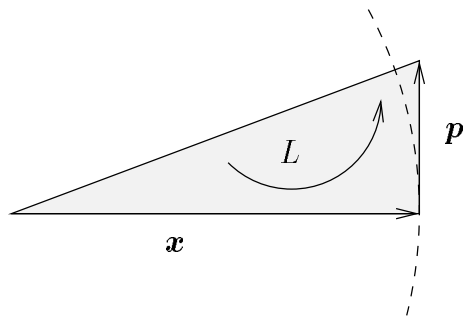


Figure 3.1: *Angular momentum.* The particle sweeps out the plane $L = \mathbf{x} \wedge \mathbf{p}$. The angular momentum should be directly related to the area swept out (*c.f.* Kepler's second law), so is naturally encoded as a bivector.

duality operation. We will therefore dispense with the traditional definition of angular momentum as an axial vector, and replace it with a bivector L , defined by

$$L = \mathbf{x} \wedge \mathbf{p}. \quad (3.2)$$

The steps involved in computing L are essentially the same as taken to compute \mathbf{l} , but the freedom now to make use of the geometric product can speed up many derivations.

If we differentiate L we obtain

$$\dot{L} = \mathbf{v} \wedge (m\mathbf{v}) + \mathbf{x} \wedge (\dot{\mathbf{p}}) = \mathbf{x} \wedge \mathbf{f}. \quad (3.3)$$

We define the torque N about the origin as the *bivector*

$$N = \mathbf{x} \wedge \mathbf{f}, \quad (3.4)$$

so that the torque and angular momentum are related by

$$\dot{L} = N. \quad (3.5)$$

The idea that the torque is represented by a bivector is also natural, as torques act over a plane. The plane in question is defined by the vector \mathbf{f} and the chosen origin, so both L and N depend on the origin. Recall also that bivectors are additive, much like vectors, so the result of applying two torques is found by adding the respective bivectors.

The angular momentum bivector can be written in an alternative way by first defining $r = |\mathbf{x}|$ and writing

$$\mathbf{x} = r\hat{\mathbf{x}}, \quad (3.6)$$

where $\hat{\mathbf{x}}^2 = 1$. We therefore have

$$\dot{\mathbf{x}} = \frac{d}{dt}(r\hat{\mathbf{x}}) = \dot{r}\hat{\mathbf{x}} + r\dot{\hat{\mathbf{x}}}, \quad (3.7)$$

so that

$$L = m\mathbf{x} \wedge (\dot{r}\hat{\mathbf{x}} + r\dot{\hat{\mathbf{x}}}) = mr\hat{\mathbf{x}} \wedge (\dot{r}\hat{\mathbf{x}} + r\dot{\hat{\mathbf{x}}}) = mr^2 \hat{\mathbf{x}} \wedge \dot{\hat{\mathbf{x}}}. \quad (3.8)$$

But since $\hat{\mathbf{x}}^2 = 1$ we must have

$$0 = \frac{d}{dt}(\hat{\mathbf{x}}^2) = 2\hat{\mathbf{x}} \cdot \dot{\hat{\mathbf{x}}}. \quad (3.9)$$

We can therefore eliminate the outer product in (3.8) and write

$$L = mr^2 \hat{\mathbf{x}} \dot{\hat{\mathbf{x}}} = -mr^2 \dot{\hat{\mathbf{x}}} \hat{\mathbf{x}}, \quad (3.10)$$

which is useful in a number of problems.

3.2 Two-Body Central Force Interactions

Suppose that two particles have positions \mathbf{x}_1 and \mathbf{x}_2 , and masses m_1 and m_2 . Newton's second law for the central force problem takes the form

$$m_1 \ddot{\mathbf{x}}_1 = \mathbf{f}, \quad m_2 \ddot{\mathbf{x}}_2 = -\mathbf{f}, \quad (3.11)$$

where \mathbf{f} is the inter-particle force. This system is reduced to an equivalent single-body problem by introducing the centre of mass vector \mathbf{X} and the relative separation vector \mathbf{x} ,

$$\mathbf{X} = \frac{1}{M}(m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2), \quad \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2, \quad (3.12)$$

where M is the total mass $m_1 + m_2$. The force equations now become

$$\mu \ddot{\mathbf{x}} = \mathbf{f}, \quad \text{and} \quad \ddot{\mathbf{X}} = 0, \quad (3.13)$$

where μ is the reduced mass $m_1 m_2 / M$.

The angular momentum about the centre of mass \mathbf{X} is given by

$$\begin{aligned} L &= m_1(\mathbf{X} - \mathbf{x}_1) \wedge (\dot{\mathbf{X}} - \dot{\mathbf{x}}_1) + m_2(\mathbf{X} - \mathbf{x}_2) \wedge (\dot{\mathbf{X}} - \dot{\mathbf{x}}_2) \\ &= -\mu \mathbf{x} \wedge (\dot{\mathbf{X}} - \dot{\mathbf{x}}_1) + \mu \mathbf{x} \wedge (\dot{\mathbf{X}} - \dot{\mathbf{x}}_2) \\ &= \mu \mathbf{x} \wedge \dot{\mathbf{x}}. \end{aligned} \quad (3.14)$$

For central forces \mathbf{f} is directed along \mathbf{x} , *i.e.* $\mathbf{f} = f\hat{\mathbf{x}}$. In this case it is simple to see that L is conserved, so the motion of the particles is confined to the L plane. The trajectory of \mathbf{x} must also sweep out area at a constant rate, since this is how L is defined. (For planetary motion this is Kepler's second law.) The magnitude of L is defined by

$$l^2 = |L|^2 = \langle L \tilde{L} \rangle = \mu^2 r^4 \langle \hat{\mathbf{x}} \dot{\hat{\mathbf{x}}} \hat{\mathbf{x}} \dot{\hat{\mathbf{x}}} \rangle = \mu^2 r^4 \dot{\hat{\mathbf{x}}}^2, \quad (3.15)$$

so that $l = \mu r^2 |\dot{\hat{\mathbf{x}}}|$. One of the factors of L is reversed to ensure that the scalar part of the product is positive.

The force is conservative and can be written as (minus) the gradient of a potential $V(r)$:

$$\mathbf{f} = f \hat{\mathbf{x}}, \quad f = -\frac{dV}{dr}. \quad (3.16)$$

The total energy is given by

$$E = \frac{\mu \dot{r}^2}{2} + \frac{l^2}{2\mu r^2} + V(r) \quad (3.17)$$

and, since the force is conservative, E is constant.

3.2.1 Inverse Square Forces

For an inverse square force we have $f = -k/r^2$ (k positive for attractive forces). The basic equation to solve is

$$\mu \ddot{\mathbf{x}} = -\frac{k}{r^2} \hat{\mathbf{x}} = -\frac{k}{r^3} \mathbf{x}. \quad (3.18)$$

This is a second-order vector differential equation, so we expect there to be two constant vectors in the solution — one for the initial position and one for the velocity. One of the conserved quantities is L . To find the other we form

$$L \dot{\mathbf{v}} = -\frac{k}{\mu r^2} L \hat{\mathbf{x}} = -k \hat{\mathbf{x}} \dot{\hat{\mathbf{x}}} \hat{\mathbf{x}} = k \dot{\hat{\mathbf{x}}}. \quad (3.19)$$

It follows that

$$\frac{d}{dt}(L \mathbf{v} - k \hat{\mathbf{x}}) = 0. \quad (3.20)$$

The motion is therefore described by the simple equation

$$L \mathbf{v} = k(\hat{\mathbf{x}} + \mathbf{e}), \quad (3.21)$$

where the *eccentricity vector* \mathbf{e} is a second vector constant of motion. This vector is also known in various contexts as the Laplace vector and as the Runge-Lenz vector. From its definition we can see that \mathbf{e} must lie in the L plane.

To find a direct equation for the trajectory we first write

$$L \mathbf{v} \mathbf{x} = L(\mathbf{v} \cdot \mathbf{x} + \mathbf{v} \wedge \mathbf{x}) = \frac{1}{\mu} L \tilde{L} + \mathbf{v} \cdot \mathbf{x} L = k(r + \mathbf{e} \cdot \mathbf{x}). \quad (3.22)$$

The scalar part of this equation gives

$$r = \frac{l^2}{k\mu(1 + \mathbf{e} \cdot \hat{\mathbf{x}})}. \quad (3.23)$$

This equation specifies a conic surface in three dimensions with symmetry axis \mathbf{e} . The surface is formed by rotating a two-dimensional conic about this axis. Since the motion takes place entirely within the L plane the motion is described by a conic.

3.2.2 Gravitational Perturbations

The eccentricity vector \mathbf{e} turns out to be extremely useful in celestial perturbation theory. This is a complex and difficult subject which has driven many important mathematical developments. Here we will briefly outline one calculation, which is the precession induced by general relativistic corrections to the inverse square law. General relativity modifies the force law so that it reads

$$\ddot{\mathbf{x}} = -\frac{GM}{r^2} \left(1 + \frac{3l^2}{\mu^2 c^2 r^2} \right) \hat{\mathbf{x}}. \quad (3.24)$$

(A subtlety is that the derivatives are now with respect to proper time, but this does not effect our reasoning.) The force is still central, so the angular momentum L is still conserved. However, the eccentricity vector is no longer constant, but instead satisfies

$$\dot{\mathbf{e}} = \frac{3l^2}{\mu^2 c^2 r^2} \dot{\hat{\mathbf{x}}}. \quad (3.25)$$

For bound orbits this gives rise to a precession of the major axis (see Fig. 3.2). The quantity of most interest is the amount \mathbf{e} changes in one orbit. To get an approximate result for this we assume that the orbit is precisely elliptical and write

$$\Delta \mathbf{e} = \int_0^T dt \frac{3l^2}{\mu^2 c^2 r^2} \dot{\hat{\mathbf{x}}} = -\frac{3l^2}{\mu^3 c^2} L \int_0^T dt \frac{\hat{\mathbf{x}}}{r^4}, \quad (3.26)$$

where T is the orbital period. Orbital averages such as these can be found tabulated in a number of books, and an exercise goes over the basic steps. The final result is that

$$\Delta \mathbf{e} = \frac{6\pi GM}{a(1-e^2)c^2} \mathbf{e} \cdot \hat{L}, \quad (3.27)$$

where $\hat{L} = L/l$, a is the semi-major axis and $e = |\mathbf{e}|$ is the eccentricity. This gives a precession of \mathbf{e} with the orientation of L , which corresponds to an advance (Fig. 3.2). For Mercury this gives rise to an advance in the perihelion of 43 arc seconds per century, which was famously explained by general relativity.

3.3 Rotating Frames and Angular Velocity

As a preliminary to an investigation of rigid body dynamics, we now study the use of rotors in describing rotating systems. Suppose that the frame of vectors $\{\mathbf{f}_k\}$ is

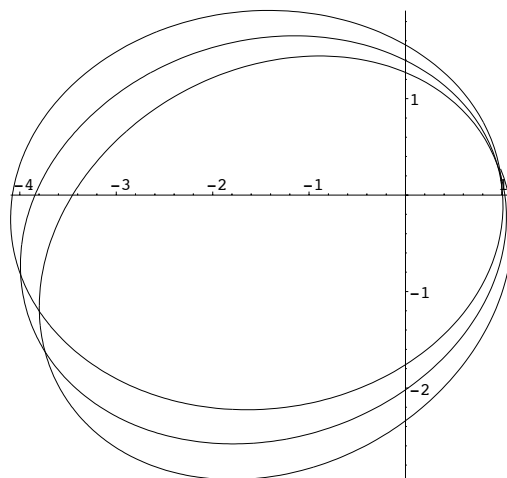


Figure 3.2: *Orbital precession.* The plot shows a modified orbit as predicted by general relativity. The ellipse precesses round in the same direction as the orbital motion. The parameters have been chosen to exaggerate the precession effect.

rotating in space. These can be related to a fixed orthonormal frame $\{e_k\}$ by the time-dependent rotor $R(t)$:

$$\mathbf{f}_k(t) = R(t)e_k\tilde{R}(t). \quad (3.28)$$

The angular velocity vector $\boldsymbol{\omega}$ is traditionally defined by the formula

$$\dot{\mathbf{f}}_k = \boldsymbol{\omega} \times \mathbf{f}_k = -I\boldsymbol{\omega} \wedge \mathbf{f}_k. \quad (3.29)$$

We can write this as the inner product between a vector and a bivector as follows:

$$\boldsymbol{\omega} \times \mathbf{f}_k = -\frac{1}{2}I(\boldsymbol{\omega} \mathbf{f}_k - \mathbf{f}_k \boldsymbol{\omega}) = \frac{1}{2}(\mathbf{f}_k I \boldsymbol{\omega} - I \boldsymbol{\omega} \mathbf{f}_k) = \mathbf{f}_k \cdot (I \boldsymbol{\omega}). \quad (3.30)$$

Since angular momentum is now viewed as a bivector, we must expect the same to be true for angular velocity. We therefore define the angular velocity bivector Ω by

$$\Omega = I\boldsymbol{\omega}. \quad (3.31)$$

This choice ensures that the rotation has the orientation implied by Ω .

3.3.1 The Rotor Equation

To see how Ω is related to the rotor R we start by differentiating equation (3.28),

$$\dot{\mathbf{f}}_k = \dot{R}e_k\tilde{R} + Re_k\dot{\tilde{R}} = \dot{R}\tilde{R}\mathbf{f}_k + \mathbf{f}_k R\dot{\tilde{R}}. \quad (3.32)$$

From the normalisation equation $R\tilde{R} = 1$ we see that

$$0 = \frac{d}{dt}(R\tilde{R}) = \dot{R}\tilde{R} + R\dot{\tilde{R}}, \quad (3.33)$$

and since differentiation and reversion are interchangeable operations we have

$$\dot{R}\tilde{R} = -R\dot{\tilde{R}} = -(\dot{R}\tilde{R})^\sim. \quad (3.34)$$

The quantity $\dot{R}\tilde{R}$ is equal to minus its own reverse and has even grade, so must be a pure bivector. The equation for \mathbf{f}_k now becomes

$$\dot{\mathbf{f}}_k = \dot{R}\tilde{R}\mathbf{f}_k - \mathbf{f}_k\dot{R}\tilde{R} = (2\dot{R}\tilde{R}) \cdot \mathbf{f}_k. \quad (3.35)$$

Comparing with equation (3.30) and equation (3.31) we see that $2\dot{R}\tilde{R}$ must equal minus the angular velocity bivector Ω , so

$$2\dot{R}\tilde{R} = -\Omega. \quad (3.36)$$

The dynamics is therefore contained in the single *rotor equation*

$$\dot{R} = -\frac{1}{2}\Omega R. \quad (3.37)$$

The reversed form of this is also useful,

$$\dot{\tilde{R}} = \frac{1}{2}\tilde{R}\Omega. \quad (3.38)$$

Equations of this type are surprisingly ubiquitous in physics. In the more general setting, rotors are viewed as elements of a *Lie group*, and the bivectors form their *Lie algebra*.

The rotor equation can be expressed in terms of either the ‘space’ Ω or ‘body’ Ω_B angular velocities. The body angular velocity is the bivector Ω rotated back to the fixed reference frame. The two are related by

$$\Omega = R\Omega_B\tilde{R}. \quad (3.39)$$

In terms of these we have

$$\dot{R} = -\frac{1}{2}\Omega R = -\frac{1}{2}R\Omega_B, \quad \text{and} \quad \dot{\tilde{R}} = \frac{1}{2}\Omega_B\tilde{R}. \quad (3.40)$$

3.3.2 Constant Ω

For the case of constant Ω equation (3.37) integrates immediately to give

$$R = e^{-\Omega t/2} R_0, \quad (3.41)$$

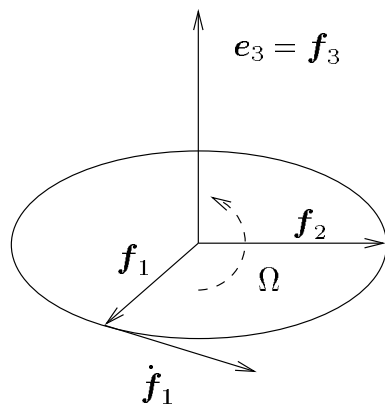


Figure 3.3: *Orientation of the angular velocity bivector.* Ω has the orientation of $\mathbf{f}_1 \wedge \dot{\mathbf{f}}_1$. It must therefore have orientation $+\mathbf{e}_1 \wedge \mathbf{e}_2$ when $\boldsymbol{\omega} = \omega \mathbf{e}_3$.

which is the rotor for a constant frequency rotation in the positive sense in the Ω plane. The frame rotates according to

$$\mathbf{f}_k(t) = e^{-\Omega t/2} R_0 \mathbf{e}_k \tilde{R}_0 e^{\Omega t/2}. \quad (3.42)$$

The constant term R_0 describes the orientation of the frame at $t = 0$, relative to the $\{\mathbf{e}_k\}$ frame.

As an example, consider the case of motion about the \mathbf{e}_3 axis (Figure 3.3). We have

$$\Omega = \omega I \mathbf{e}_3 = \omega \mathbf{e}_1 \mathbf{e}_2, \quad (3.43)$$

and for convenience we set $R_0 = 1$. The motion is described by

$$\mathbf{f}_k(t) = \exp\left(-\frac{1}{2} \mathbf{e}_1 \mathbf{e}_2 \omega t\right) \mathbf{e}_k \exp\left(\frac{1}{2} \mathbf{e}_1 \mathbf{e}_2 \omega t\right), \quad (3.44)$$

so that the \mathbf{f}_1 axis rotates as

$$\mathbf{f}_1 = \mathbf{e}_1 \exp(\mathbf{e}_1 \mathbf{e}_2 \omega t) = \cos(\omega t) \mathbf{e}_1 + \sin(\omega t) \mathbf{e}_2. \quad (3.45)$$

This defines a *right-handed* (anticlockwise) rotation in the $\mathbf{e}_1 \mathbf{e}_2$ plane, as prescribed by the orientation of Ω .

3.4 Rigid Body Dynamics

A rigid body can be viewed as a system of particles moving subject to the constraint that all inter-particle distances are fixed. Such a system can be described by two vector

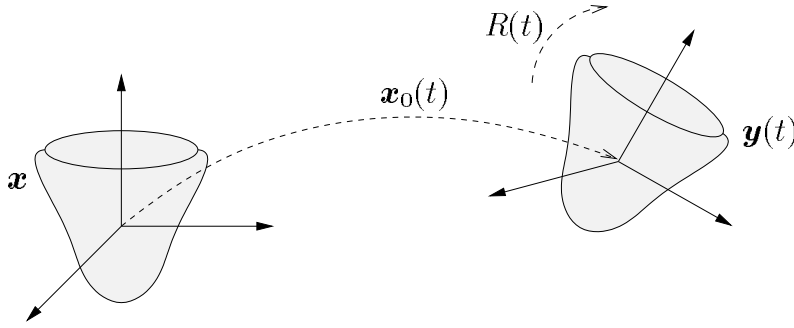


Figure 3.4: *Description of a rigid body.* The vector $\mathbf{x}_0(t)$ specifies the position of the centre of mass, relative to the origin. The rotor $R(t)$ defines the orientation of the body, relative to a fixed copy imagined placed at the origin. \mathbf{x} is a vector in the reference body, and \mathbf{y} is the vector in space of the equivalent point on the moving body.

variables, one for position of the centre of mass, and one for the vector from the centre of mass to the point in the body. The latter can be encoded in terms of a rotation from a fixed ‘reference’ body onto the body in space (Figure 3.4). We let \mathbf{x}_0 denote the position of the centre of mass and $\mathbf{y}_i(t)$ denote the position (in space) of a point in the body. These are related by

$$\mathbf{y}_i(t) = R(t)\mathbf{x}_i\tilde{R}(t) + \mathbf{x}_0(t), \quad (3.46)$$

where \mathbf{x}_i is a fixed constant vector in the reference copy of the body. In this manner we have placed all of the rotational motion in the time-dependent rotor $R(t)$.

The velocity of the point $\mathbf{y} = R\mathbf{x}\tilde{R} + \mathbf{x}_0$ is

$$\begin{aligned} \mathbf{v}(t) &= \dot{R}\mathbf{x}\tilde{R} + R\mathbf{x}\dot{\tilde{R}} + \dot{\mathbf{x}}_0 \\ &= -\frac{1}{2}R\Omega_B\mathbf{x}\tilde{R} + \frac{1}{2}R\mathbf{x}\Omega_B\tilde{R} + \mathbf{v}_0 \\ &= R\mathbf{x}\cdot\Omega_B\tilde{R} + \mathbf{v}_0 \end{aligned} \quad (3.47)$$

where \mathbf{v}_0 is the velocity of the centre of mass. Note that we have suppressed unnecessary brackets to write $R\mathbf{x}\cdot\Omega_B\tilde{R}$ in place of $R(\mathbf{x}\cdot\Omega_B)\tilde{R}$. We have also chosen to work in terms of the ‘body’ angular velocity Ω_B , for reasons which will emerge shortly.

To simplify calculations we employ a continuum approximation and introduce a density $\rho = \rho(\mathbf{x})$. The position vector \mathbf{x} is taken relative to the centre of mass, so we have

$$\int d^3x \rho = M, \quad \text{and} \quad \int d^3x \rho \mathbf{x} = 0. \quad (3.48)$$

The momentum of the rigid body is simply

$$\int d^3x \rho \mathbf{v} = \int d^3x \rho (R\mathbf{x}\cdot\Omega_B\tilde{R} + \mathbf{v}_0) = M\mathbf{v}_0, \quad (3.49)$$

so is specified entirely by the motion of the centre of mass.

3.4.1 The Inertia Tensor

The next quantity we require is the angular momentum bivector L for the body about its centre of mass. We therefore form

$$\begin{aligned}
 L &= \int d^3x \rho(\mathbf{y} - \mathbf{x}_0) \wedge \mathbf{v} \\
 &= \int d^3x \rho(R\mathbf{x}\tilde{R}) \wedge (R\mathbf{x} \cdot \Omega_B \tilde{R} + \mathbf{v}_0) \\
 &= R \left(\int d^3x \rho \mathbf{x} \wedge (\mathbf{x} \cdot \Omega_B) \right) \tilde{R}.
 \end{aligned} \tag{3.50}$$

The integral inside the brackets refers only to the fixed copy and so defines a time-independent function of Ω_B . This is the reason for working with Ω_B instead of the space angular velocity Ω . We define the *inertia tensor* $\mathcal{I}(B)$ by

$$\mathcal{I}(B) = \int d^3x \rho \mathbf{x} \wedge (\mathbf{x} \cdot B). \tag{3.51}$$

This is a linear function mapping bivectors to bivectors. This type of notation for linear functions is the natural extension of the index-free approach followed throughout this course. The inertia tensor has following two properties, holding for any bivectors A and B :

$$\begin{aligned}
 \text{Linearity:} & \quad \mathcal{I}(\lambda A + \mu B) = \lambda \mathcal{I}(A) + \mu \mathcal{I}(B) \\
 \text{Symmetry:} & \quad \langle A \mathcal{I}(B) \rangle = \langle \mathcal{I}(A) B \rangle.
 \end{aligned} \tag{3.52}$$

Verifying these properties is left as an exercise.

The fact that the inertia tensor maps bivectors to bivectors, rather than vectors to vectors, is a further break from tradition. Since both vectors and bivectors belong to a three-dimensional linear space, there is no additional complexity introduced in this new picture. To understand the effect of the inertia tensor, suppose that as input it receives the bivector B . We imagine what would happen if the body were to rotate about the centre of mass in the B plane. The point \mathbf{x} would move with velocity $\mathbf{x} \cdot B$, and the momentum density at this point would be $\rho \mathbf{x} \cdot B$, as shown in Figure 3.5. The angular momentum density bivector would therefore be $\mathbf{x} \wedge (\rho \mathbf{x} \cdot B)$, and integrating this over the entire body returns the total angular momentum bivector for rotation in the B plane. In general, the result of this process will not lie in the same plane as the input bivector B . As the motion proceeds the angular velocity Ω is back-rotated to the body system, and $\Omega_B = \tilde{R}\Omega R$ is fed into the (fixed) inertia tensor. The result of this, $\mathcal{I}(\Omega_B)$, is then rotated onto the space angular momentum L .

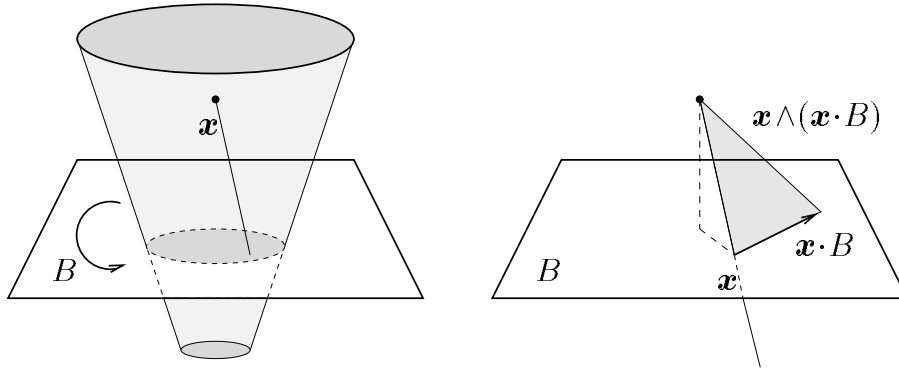


Figure 3.5: *The Inertia Tensor*. The inertia tensor $\mathcal{I}(B)$ is a linear function mapping its bivector argument B onto a bivector. It returns the total angular momentum about the centre of mass for rotation in the B plane.

3.4.2 Equations of Motion

The inertia tensor is constructed from the point of view of the fixed body. From equation (3.50) we see that the angular momentum in space is obtained by rotating the body angular momentum $\mathcal{I}(\Omega_B)$ onto the space configuration, that is

$$L = R\mathcal{I}(\Omega_B)\tilde{R}. \quad (3.53)$$

The equations of motion are $\dot{L} = N$, where N is the external torque. The inertia tensor is time-independent since it only refers to the static ‘reference’ copy of the rigid body, so we find that

$$\begin{aligned} \dot{L} &= \dot{R}\mathcal{I}(\Omega_B)\tilde{R} + R\mathcal{I}(\Omega_B)\dot{\tilde{R}} + R\mathcal{I}(\dot{\Omega}_B)\tilde{R} \\ &= R[\mathcal{I}(\dot{\Omega}_B) - \frac{1}{2}\Omega_B\mathcal{I}(\Omega_B) + \frac{1}{2}\mathcal{I}(\Omega_B)\Omega_B]\tilde{R} \\ &= R[\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B)]\tilde{R}. \end{aligned} \quad (3.54)$$

(Recall that the commutator of the bivectors Ω_B and $\mathcal{I}(\Omega_B)$ must result in a third bivector.)

The equations are more commonly seen in component form. To recover these we introduce a set of *principal axes* $\{\mathbf{e}_k\}$. These satisfy

$$\mathcal{I}(I\mathbf{e}_k) = i_k I\mathbf{e}_k \quad \text{no sum}, \quad (3.55)$$

where the i_k are the 3 principal moments of inertia. The symmetry of $\mathcal{I}(B)$ ensures that it is always possible to find a set of principal axes. In terms of these we write

$$\Omega = \sum_{k=1}^3 \omega_k I\mathbf{f}_k, \quad \Omega_B = \sum_{k=1}^3 \omega_k I\mathbf{e}_k, \quad (3.56)$$

and

$$L = \sum_{k=1}^3 i_k \omega_k I \mathbf{f}_k, \quad N = \sum_{k=1}^3 N_k I \mathbf{f}_k. \quad (3.57)$$

Expanding out in terms of components we recover the Euler equations of motion for a rigid body (exercise)

$$\begin{aligned} i_1 \dot{\omega}_1 - \omega_2 \omega_3 (i_2 - i_3) &= N_1 \\ i_2 \dot{\omega}_2 - \omega_3 \omega_1 (i_3 - i_1) &= N_2 \\ i_3 \dot{\omega}_3 - \omega_1 \omega_2 (i_1 - i_2) &= N_3. \end{aligned} \quad (3.58)$$

3.4.3 Torque-free motion

The torque-free equation $\dot{L} = 0$ reduces to

$$\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B) = 0. \quad (3.59)$$

This is a first-order constant coefficient differential equation for the bivector Ω_B . Closed form solutions exist, but can be messy. Useful insights are obtained by considering the conserved quantities in the problem. The first of these is the angular momentum L . A second is provided by the rotational component of the kinetic energy, which is

$$T = \frac{1}{2} \int d^3x \rho (R \mathbf{x} \cdot \Omega_B \tilde{R})^2 = \frac{1}{2} \int d^3x \rho (\mathbf{x} \cdot \Omega_B)^2. \quad (3.60)$$

If we now employ the rearrangement

$$(\mathbf{x} \cdot \Omega_B)^2 = \langle \mathbf{x} \cdot \Omega_B \mathbf{x} \Omega_B \rangle = -\Omega_B \cdot (\mathbf{x} \wedge (\mathbf{x} \cdot \Omega_B)) \quad (3.61)$$

we see that the rotational energy is

$$T = -\frac{1}{2} \Omega_B \cdot \mathcal{I}(\Omega_B) = \frac{1}{2} \tilde{\Omega}_B \cdot \mathcal{I}(\Omega_B). \quad (3.62)$$

It is not hard to show that this is also conserved.

We now introduce the components $L_k = i_k \omega_k$, so that

$$L = \sum_{k=1}^3 L_k I \mathbf{f}_k. \quad (3.63)$$

The L_k are the components of L in the rotating \mathbf{f}_k frame. So, even though L is constant, the components are time-dependent. In terms of these components the magnitude of L is

$$L\tilde{L} = L_1^2 + L_2^2 + L_3^2 \quad (3.64)$$

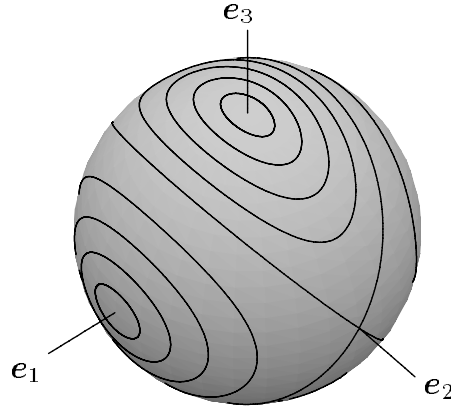


Figure 3.6: *Angular momentum Orbits*. The point described by the vector \mathbf{l} simultaneously lies on the surface of a sphere and an ellipsoid. The figure shows possible paths on the sphere for \mathbf{l} in the case of $i_1 < i_2 < i_3$, with the 3-axis vertical.

and the kinetic energy is

$$T = \frac{L_1^2}{2i_1} + \frac{L_2^2}{2i_2} + \frac{L_3^2}{2i_3}. \quad (3.65)$$

Both $|L|$ and T are constants of motion, which imposes two constraints on the three components L_k . A useful way to visualise this is to think in terms of a vector \mathbf{l} with components L_k ,

$$\mathbf{l} = \sum_{k=1}^3 L_k \mathbf{e}_k = -I \tilde{R} L R. \quad (3.66)$$

This is the vector perpendicular to $\tilde{R} L R$ — a rotating vector in the fixed reference body. Conservation of $|L|$ means that \mathbf{l} is constrained to lie on a sphere, and conservation of T restricts \mathbf{l} to the surface of an ellipsoid. Possible paths for \mathbf{l} for a given rigid body are therefore defined by the intersections of a sphere with a family of ellipsoids (governed by T). For the case of unequal principal moments these orbits are non-degenerate. Examples of these orbits are shown in figure 3.6. This figure shows that orbits around the axes with the smallest and largest principal moments are stable, whereas around the middle axis the orbits are unstable. Any small change in the energy of the body will tend to throw it into a very different orbit if the orbit of \mathbf{l} takes it near \mathbf{e}_2 .

3.4.4 Example — The Symmetric Top

If the body has a single symmetry axis, the solution for torque-free motion is quite straightforward in the rotor framework. In this case the body has two equal moments of inertia, $i_1 = i_2$, and the third principal moment i_3 is assumed to be different. With

this assignment \mathbf{e}_3 is the symmetry axis of the body. The action of the inertia tensor on Ω_B is

$$\begin{aligned}\mathcal{I}(\Omega_B) &= i_1\omega_1\mathbf{e}_2\mathbf{e}_3 + i_1\omega_2\mathbf{e}_3\mathbf{e}_1 + i_3\omega_3\mathbf{e}_1\mathbf{e}_2 \\ &= i_1\Omega_B + (i_3 - i_1)\omega_3 I\mathbf{e}_3,\end{aligned}\quad (3.67)$$

and the final Euler equation in (3.58) tells us that that ω_3 is a constant. Rotating both sides of this equation with R now yields

$$\Omega = R\Omega_B\tilde{R} = \frac{1}{i_1}L + \frac{i_1 - i_3}{i_1}\omega_3 R I\mathbf{e}_3\tilde{R}. \quad (3.68)$$

Our rotor equation therefore becomes

$$\dot{R} = -\frac{1}{2}\Omega R = -\frac{1}{2i_1}(LR + R(i_1 - i_3)\omega_3 I\mathbf{e}_3). \quad (3.69)$$

The right-hand side of this equation involves two constant bivectors, one multiplying R to the left and the other to the right. We therefore define the two bivectors

$$\Omega_l = \frac{1}{i_1}L, \quad \Omega_r = \omega_3 \frac{i_1 - i_3}{i_1} I\mathbf{e}_3, \quad (3.70)$$

so that the rotor equation becomes

$$\dot{R} = -\frac{1}{2}\Omega_l R - \frac{1}{2}R\Omega_r. \quad (3.71)$$

This equation integrates immediately to give

$$R(t) = \exp(-\frac{1}{2}\Omega_l t)R_0 \exp(-\frac{1}{2}\Omega_r t). \quad (3.72)$$

This fully describes the motion of a symmetric top. It shows that there is an ‘internal’ rotation in the $\mathbf{e}_1\mathbf{e}_2$ plane (the symmetry plane of the body). This is responsible for the precession of a symmetric top. The constant rotor R_0 defines the attitude of the rigid body at $t = 0$ and can be set to 1. The resultant body is then rotated in the plane of its angular momentum to obtain the final attitude in space.

Chapter 4

Quantum Mechanics

The first major application of a geometric algebra in physics was in the theory of quantum spin. The Pauli matrix algebra forms a 2×2 representation of the geometric algebra of 3-d space. (We usually refer to such a matrix algebra as a Clifford algebra.) It is no surprise then that much of quantum theory finds a natural expression within geometric algebra. To see this, however, one must reconsider the standard interpretation of the quantum spin operators. Like much discussion of the interpretation of quantum theory, the issues raised here are controversial. There is no question about the validity of this new algebraic approach, however, and little doubt about its advantages. Whether the algebraic simplifications obtained here are indicative of a deeper structure embedded in quantum mechanics is an open question.

4.1 Non-Relativistic Quantum Spin

The Stern-Gerlach experiment was the first to demonstrate the quantum nature of the magnetic moment. In these experiments, a beam of particles passes through a non-uniform magnetic field \mathbf{B} . Classically, one would expect the force on each particle to be governed by the equation

$$\mathbf{f} = \boldsymbol{\mu} \cdot \nabla \mathbf{B}, \quad (4.1)$$

where $\boldsymbol{\mu}$ is the magnetic moment. This would give rise to a continuous distribution after passing through the field. Instead, what is observed is a number of evenly-spaced discrete bands (Fig. 4.1). The magnetic moment is quantised in the same way as the angular momentum.

When silver atoms are used to make up the beam there is a further surprise: only two beams emerge on the far side. Silver atoms contain a single electron in their outermost shell, so it looks as if electrons have an intrinsic angular momentum which can take only

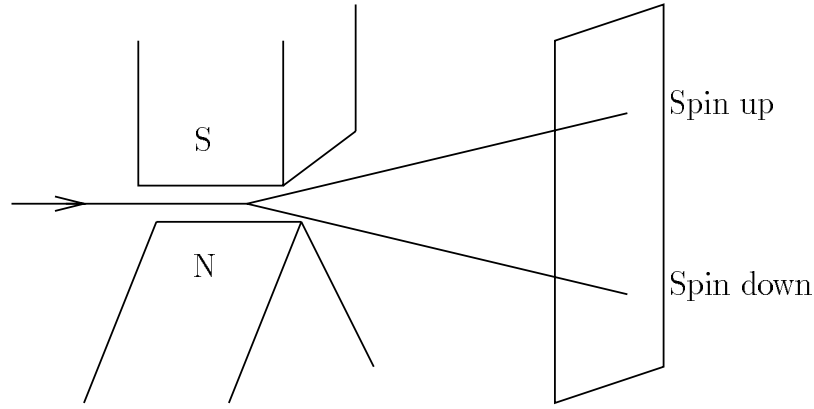


Figure 4.1: *A Stern-Gerlach Experiment.* A particle beam is sent through a highly non-uniform \mathbf{B} -field. What emerges is a set of discrete, evenly spaced beams.

two values. This is known as its *spin*, though no classical picture should be inferred from this name. The double-valued nature of the spin suggests that the electron's wavefunction should contain two terms, representing a superposition of the possible spin states,

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \quad (4.2)$$

where α and β are complex numbers. Such a state can be represented in matrix form as the *spinor*

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \quad (4.3)$$

If we align the z -axis with the spin-up direction, then the operator returning the spin along the z -axis must be

$$\hat{s}_3 = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.4)$$

where λ is to be determined. The spin is added to the orbital angular momentum to give a conserved total angular momentum operator $\hat{\mathbf{j}} = \hat{\mathbf{l}} + \hat{\mathbf{s}}$. For this to make sense the spin operators should have the same commutation relations as the angular momentum operators \hat{l}_i ,

$$\hat{l}_i = -i\hbar\epsilon_{ijk}x_j\partial_k, \quad [\hat{l}_i, \hat{l}_j] = i\hbar\epsilon_{ijk}\hat{l}_k. \quad (4.5)$$

An exercise shows how to find the remaining operators. The result is that the spin operators are given by

$$\hat{s}_k = \frac{1}{2}\hbar\hat{\sigma}_k, \quad (4.6)$$

where the $\hat{\sigma}_k$ are the *Pauli matrices*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.7)$$

The Pauli matrices satisfy the commutation relations,

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k. \quad (4.8)$$

They also have the property that two different matrices *anticommute*,

$$\hat{\sigma}_1\hat{\sigma}_2 + \hat{\sigma}_2\hat{\sigma}_1 = 0, \quad \text{etc.}, \quad (4.9)$$

and all of the matrices square to the identity matrix,

$$\hat{\sigma}_1^2 = \hat{\sigma}_2^2 = \hat{\sigma}_3^2 = 1. \quad (4.10)$$

These are the defining relations for a set of orthonormal basis vectors in the geometric algebra \mathcal{G}_3 ! The Pauli matrices are a *matrix representation* of the algebra. Any multivector in \mathcal{G}_3 can be written as a 2×2 complex matrix. Scalars are replaced by multiples of the identity matrix, and the pseudoscalar I is replaced by i times the identity. The set of 2×2 complex matrices is a linear space of real dimension 8, the same as the dimension of \mathcal{G}_3 . Any multivector product can be computed by first constructing matrix equivalents for the multivectors and multiplying the matrices. (Matrix multiplication is associative, like the geometric product.) But, while matrix representations can be useful, most manipulations in geometric algebra are more efficiently performed without their introduction.

4.2 Spinors and Multivectors

Given that the Pauli matrices are a representation of \mathcal{G}_3 , can we give a more geometric picture to the act of forming spinor observables? The answer is yes, but first we need to represent the spinor (4.3) within the geometric algebra of space. To find a natural way to do this we consider the observables of a spinor. These are the eigenvalues of Hermitian operators and, for 2 state systems, the relevant operators are the Pauli matrices. We therefore form the three observables

$$s_k = \frac{1}{2}\hbar n_k = \langle \psi | \hat{s}_k | \psi \rangle. \quad (4.11)$$

These are the components of a single vector in the quantum theory of spin. The factor of $\hbar/2$ is irrelevant and we focus attention on the components

$$\begin{aligned} n_1 &= \langle \psi | \hat{\sigma}_1 | \psi \rangle = \alpha\beta^* + \alpha^*\beta \\ n_2 &= \langle \psi | \hat{\sigma}_2 | \psi \rangle = i(\alpha\beta^* - \alpha^*\beta) \\ n_3 &= \langle \psi | \hat{\sigma}_3 | \psi \rangle = \alpha\alpha^* - \beta\beta^*. \end{aligned} \quad (4.12)$$

The magnitude of the vector with components n_k is

$$\begin{aligned} |\mathbf{n}|^2 &= (\alpha\beta^* + \alpha^*\beta) - (\alpha\beta^* - \alpha^*\beta)^2 + (\alpha\alpha^* - \beta\beta^*)^2 \\ &= (|\alpha|^2 + |\beta|^2)^2 = \langle \psi | \psi \rangle^2. \end{aligned} \quad (4.13)$$

So, provided the state is normalised to 1, the vector \mathbf{n} must have unit length. We can therefore introduce polar coordinates and write

$$\begin{aligned} n_1 &= \sin(\theta) \cos(\phi) \\ n_2 &= \sin(\theta) \sin(\phi) \\ n_3 &= \cos(\theta). \end{aligned} \quad (4.14)$$

Comparing with Eq. (4.13) we see that we must have

$$\alpha = \cos(\theta/2) e^{i\gamma}, \quad \beta = \sin(\theta/2) e^{i\delta} \quad (4.15)$$

where $\delta - \gamma = \phi$. It follows that the spinor can be written in terms of the polar coordinates of the vector observable as

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix} e^{i(\gamma + \delta)/2}. \quad (4.16)$$

The overall phase factor can be ignored, and what remains is a description in terms of half angles. This suggests a strong analogy with *rotors*. To investigate this analogy, we recall that the polar coordinates can be viewed as part of an instruction to rotate the 3-axis onto the chosen vector. To expose this we write the vector \mathbf{n} as

$$\mathbf{n} = \sin(\theta) (\cos(\phi) \boldsymbol{\sigma}_1 + \sin(\phi) \boldsymbol{\sigma}_2) + \cos(\theta) \boldsymbol{\sigma}_3, \quad (4.17)$$

where we use $\{\boldsymbol{\sigma}_k\}$ as our orthonormal frame. These are the vector equivalents of the Pauli matrix operators. We now write

$$\begin{aligned} \mathbf{n} &= \sin(\theta) e^{-\phi I \boldsymbol{\sigma}_3 / 2} \boldsymbol{\sigma}_1 e^{\phi I \boldsymbol{\sigma}_3 / 2} + \cos(\theta) \boldsymbol{\sigma}_3 \\ &= e^{-\phi I \boldsymbol{\sigma}_3 / 2} (\cos(\theta) \boldsymbol{\sigma}_3 + \sin(\theta) \boldsymbol{\sigma}_1) e^{\phi I \boldsymbol{\sigma}_3 / 2} \\ &= e^{-\phi I \boldsymbol{\sigma}_3 / 2} e^{-\theta I \boldsymbol{\sigma}_2 / 2} \boldsymbol{\sigma}_3 e^{\theta I \boldsymbol{\sigma}_2 / 2} e^{\phi I \boldsymbol{\sigma}_3 / 2} \\ &= R \boldsymbol{\sigma}_3 \tilde{R} \end{aligned} \quad (4.18)$$

where

$$R = e^{-\phi I \boldsymbol{\sigma}_3 / 2} e^{-\theta I \boldsymbol{\sigma}_2 / 2}. \quad (4.19)$$

This suggests that there should be a natural map between the normalised spinor (4.16) and the rotor R . Both belong to vector spaces of real dimension 4 and both are normalised. Expanding out the rotor R the following one-to-one map suggests itself:

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k I \boldsymbol{\sigma}_k. \quad (4.20)$$

This map will enable us to perform all operations involving spinors without leaving the geometric algebra of space. This removes one of the conceptual gaps between classical and quantum theories. On this scheme the spin-up and spin-down basis states $|\uparrow\rangle$ and $|\downarrow\rangle$ become

$$|\uparrow\rangle \leftrightarrow 1 \quad |\downarrow\rangle \leftrightarrow -I\sigma_2. \quad (4.21)$$

One can immediately see for these that the vectors of observables have components $(0, 0, \pm 1)$, as required.

4.2.1 Pauli Operators

The representation of (4.20) can only be useful if the action of the quantum operators $\{\hat{\sigma}_k\}$ has a sensible analog. To see what this is consider, for example

$$\hat{\sigma}_1|\psi\rangle = \begin{pmatrix} -a^2 + ia^1 \\ a^0 + ia^3 \end{pmatrix} \leftrightarrow -a^2 + a^1 I\sigma_3 - a^0 I\sigma_2 + a^3 I\sigma_1. \quad (4.22)$$

It is not hard to show that the right-hand side is equal to

$$\sigma_1(a^0 + a^1 I\sigma_1 + a^2 I\sigma_2 + a^3 I\sigma_3)\sigma_3 = \sigma_1\psi\sigma_3. \quad (4.23)$$

The same is true for the remaining matrices and we can establish the map

$$\hat{\sigma}_k|\psi\rangle \leftrightarrow \sigma_k\psi\sigma_3 \quad (k = 1, 2, 3). \quad (4.24)$$

The σ_3 on the right-hand side ensures that the multivector remains in the even subalgebra. The choice of vector here reflects the choice of matrix representation, and does not break rotational invariance.

Now that we have a translation for the action of the Pauli matrices, we can find the equivalent of multiplying by the unit imaginary i . To find this we note that

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad (4.25)$$

so multiplication of both components of $|\psi\rangle$ by i can be achieved by multiplying by the product of the three matrix operators. We therefore arrive at the translation

$$i|\psi\rangle \leftrightarrow \sigma_1\sigma_2\sigma_3\psi(\sigma_3)^3 = \psi I\sigma_3. \quad (4.26)$$

So on this scheme the unit imaginary of quantum theory is replaced by right multiplication by the bivector $I\sigma_3$. This is very suggestive, though it should be borne in mind that this conclusion is a feature of our chosen representation. The appearance of the bivector $I\sigma_3$ is to be expected, since the vector of observables was formed by rotating the σ_3 vector. This vector is unchanged by rotations in the $I\sigma_3$ plane, which provides a geometric picture of phase invariance.

4.2.2 Pauli Observables

We next need to establish the quantum inner product for our multivector forms of spinors. We first note that the Hermitian adjoint operation has $\hat{\sigma}_k^\dagger = \hat{\sigma}_k$, and reverses the order of all products. This is precisely the same as the reversion operation for multivectors, and the two can be used interchangeably:

$$M^\dagger = \tilde{M}. \quad (4.27)$$

The reason why we did not use the dagger symbol for reversion from the outset is that this is incompatible with current usage in relativistic quantum theory. There the Hermitian adjoint is a frame-dependent operation, whereas reversion is independent of frame.

The quantum inner product is

$$\langle \psi | \phi \rangle = (\psi_1^*, \psi_2^*) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \psi_1^* \phi_1 + \psi_2^* \phi_2, \quad (4.28)$$

where we ignore spatial integrals. For a wide range of problems the spatial and spin components of the wave function can be separated. If this is not the case then the quantum inner product should also contain an integral over all space. The result of the real part (denoted by \mathbf{R}) of the inner product is reproduced by

$$\mathbf{R}\langle \psi | \phi \rangle \leftrightarrow \langle \tilde{\psi} | \phi \rangle, \quad (4.29)$$

so that, for example,

$$\langle \psi | \psi \rangle \leftrightarrow \langle \tilde{\psi} | \psi \rangle = \langle (a^0 - a^j I \sigma_j)(a^0 + a^k I \sigma_k) \rangle = (a^0)^2 + a^k a^k. \quad (4.30)$$

Since

$$\langle \psi | \phi \rangle = \mathbf{R}\langle \psi | \phi \rangle - i \mathbf{R}\langle \psi | i \phi \rangle, \quad (4.31)$$

the full inner product can be written

$$\langle \psi | \phi \rangle \leftrightarrow \langle \tilde{\psi} | \phi \rangle - \langle \tilde{\psi} | \phi I \sigma_3 \rangle I \sigma_3. \quad (4.32)$$

The right hand side projects out the 1 and $I \sigma_3$ components from the geometric product $\tilde{\psi} \phi$. The result of this projection on a multivector A is written $\langle A \rangle_q$. For even grade multivectors in 3-d this projection has the simple form

$$\langle A \rangle_q = \frac{1}{2}(A + \sigma_3 A \sigma_3). \quad (4.33)$$

If the result of an inner product is used to multiply a second multivector, one has to remember to keep the terms in $I \sigma_3$ to the right of the multivector. This might appear a slightly clumsy procedure at first, but it is easy to establish conventions so that manipulations are just as efficient as in the standard treatment. Furthermore, the fact that all manipulations are now performed within the geometric algebra framework offers a number of new ways to simplify the analysis of many problems.

4.2.3 The Spin Vector

As a check on the consistency of our scheme, we return to the expectation value of the spin in the k -direction, $\langle \psi | \hat{s}_k | \psi \rangle$. For this we require

$$\langle \psi | \hat{\sigma}_k | \psi \rangle \leftrightarrow \langle \tilde{\psi} \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \rangle - \langle \tilde{\psi} \boldsymbol{\sigma}_k \psi I \rangle I \boldsymbol{\sigma}_3. \quad (4.34)$$

Since $\tilde{\psi} I \boldsymbol{\sigma}_k \psi$ reverses to give minus itself, it has zero scalar part and the final term on the right-hand side vanishes. This is to be expected, as the $\hat{\sigma}_k$ are Hermitian operators. For the remaining term we note that in 3-d $\psi \boldsymbol{\sigma}_3 \tilde{\psi}$ is both odd grade and reverses to itself, so is a pure vector. We therefore define the *spin vector*

$$\mathbf{s} = \frac{1}{2} \hbar \psi \boldsymbol{\sigma}_3 \tilde{\psi}. \quad (4.35)$$

The quantum expectation now reduces to

$$\langle \psi | \hat{s}_k | \psi \rangle = \frac{1}{2} \hbar \langle \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \tilde{\psi} \rangle = \boldsymbol{\sigma}_k \cdot \mathbf{s}. \quad (4.36)$$

This new expression has a rather different interpretation to that usually encountered in quantum theory. Rather than forming the expectation value of a quantum operator, we are simply projecting out the k th component of the vector \mathbf{s} .

The rotor form defined earlier is recovered by first defining the scalar

$$\rho \equiv \psi \tilde{\psi}. \quad (4.37)$$

The spinor ψ then decomposes into

$$\psi = \rho^{1/2} R, \quad (4.38)$$

where $R = \rho^{-1/2} \psi$. The multivector R satisfies $R \tilde{R} = 1$, so is a rotor. In this approach, Pauli spinors are simply unnormalised rotors! The spin-vector \mathbf{s} can now be written as

$$\mathbf{s} = \frac{1}{2} \hbar \rho R \boldsymbol{\sigma}_3 \tilde{R}, \quad (4.39)$$

which recovers the form of Eq. (4.18).

The double-sided construction of the expectation value of Eq. (4.34) contains an instruction to rotate the fixed $\boldsymbol{\sigma}_3$ axis into the spin direction and dilate it, which is how the multivector version was established. It might appear here that we are singling out some preferred direction in space. But in fact all we are doing is borrowing a familiar idea from rigid-body dynamics. The $\boldsymbol{\sigma}_3$ on the right of ψ represents a vector in a ‘reference’ frame. All physical vectors, like \mathbf{s} , are obtained by rotating this frame onto the physical value (see Fig. 4.2). There is nothing special about $\boldsymbol{\sigma}_3$ — one can choose any (constant) reference frame and use the appropriate rotation onto \mathbf{s} , in the same way that there is nothing special about the orientation of the reference configuration of a rigid body. In rigid-body mechanics this freedom is usually employed to align the reference configuration with the initial state of the body. In quantum theory the convention is to work with the z -axis as the reference vector.

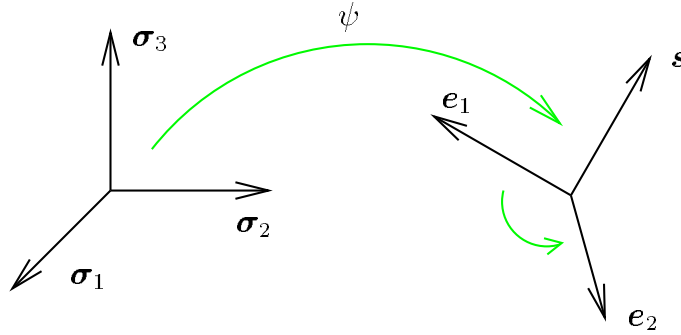


Figure 4.2: *The Spin Vector*. The normalised spinor ψ transforms the initial, reference frame onto the frame $\{e_k\}$. The vector e_3 is the spin vector. A phase transformation of ψ generates a rotation in the e_1e_2 plane. Such a transformation is unobservable, so the e_1 and e_2 vectors are also unobservable.

4.2.4 Rotating Spinors

Suppose that the vector \mathbf{s} is to be rotated to a new vector $R_0\mathbf{s}\tilde{R}_0$. The spinor ψ must transform according to

$$\psi \mapsto R_0\psi. \quad (4.40)$$

Now suppose that for R_0 we use the rotor R_θ ,

$$R_\theta = \exp(-\hat{B}\theta/2), \quad (4.41)$$

where $\hat{B}^2 = -1$ is a constant bivector. The resulting spinor is

$$\psi' = R_\theta\psi = e^{-\hat{B}\theta/2}\psi. \quad (4.42)$$

We now start to increase θ from 0 through to 2π , so that $\theta = 2\pi$ corresponds to a 360° rotation, *i.e.* the identity. But under this we see that ψ transforms to

$$\psi' = e^{-\hat{B}\pi}\psi = (\cos \pi - \hat{B} \sin \pi)\psi = -\psi. \quad (4.43)$$

So spinors change sign under 360° rotations. If a spin vector is rotated through 360° , the wavefunction does not come back to itself, but instead changes sign. This is the distinguishing property of spin-1/2 fermions in quantum theory. Once one sees the rotor derivation of this result, however, it is rather less mysterious. Indeed, there are classical phenomena involving systems of linked rotations that show precisely the same property. (Think of the 4π symmetry observed when rotating an arm holding a tray).

4.3 Application — Magnetic Fields

Particles with non-zero spin also have a magnetic moment which is proportional to the spin. This is expressed as the operator relation

$$\hat{\mu}_k = \gamma \hat{s}_k, \quad (4.44)$$

where $\hat{\mu}_k$ is the magnetic moment operator, γ is the gyromagnetic ratio and \hat{s}_k is the spin operator. The gyromagnetic ratio is usually written in the form

$$\gamma = g \frac{q}{2m}, \quad (4.45)$$

where m is the particle mass, q is the charge and g is the reduced gyromagnetic ratio. The latter are determined experimentally to be

electron	$g_e = 2$	(actually $2(1 + \alpha/2\pi + \dots)$)
proton	$g_p = 5.587$	
neutron	$g_n = -3.826$	(use proton charge)

The value for the neutron is negative because its spin and magnetic moment are anti-parallel. All of the above are spin-1/2 particles for which we have $\hat{s}_k = \frac{1}{2}\hbar\hat{\sigma}_k$.

4.3.1 Particle in a Magnetic Field

Now suppose that the particle is placed in a magnetic field, and that all of the spatial dynamics has been separated out. We introduce the Hamiltonian operator

$$\hat{H} = -\frac{1}{2}\gamma\hbar B_k \hat{\sigma}_k = -\hat{\mu}_k B_k. \quad (4.46)$$

The spin state at time t is then written as

$$|\psi(t)\rangle = \alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle, \quad (4.47)$$

with α and β general complex coefficients. The dynamical equation for these coefficients is given by the time-dependent Schrödinger equation

$$\hat{H}|\psi\rangle = i\hbar \frac{d|\psi\rangle}{dt}. \quad (4.48)$$

This is conventionally hard to analyse, because one ends up with a pair of coupled differential equations in α and β .

Now let us see what the Schrödinger equation looks like in our new setup. We first write the equation in the form

$$\frac{d|\psi\rangle}{dt} = \frac{1}{2}\gamma i B_k \hat{\sigma}_k |\psi\rangle. \quad (4.49)$$

Now replacing $|\psi\rangle$ by the multivector ψ we see that the left-hand side is simply $\dot{\psi}$, where the dot denotes the time derivative. The right-hand side involves multiplication of the spinor $|\psi\rangle$ by $i\hat{\sigma}_k$, which we replace by

$$i\hat{\sigma}_k |\psi\rangle \leftrightarrow \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 (I\boldsymbol{\sigma}_3) = I\boldsymbol{\sigma}_k \psi. \quad (4.50)$$

The Schrödinger equation (4.48) is therefore simply

$$\dot{\psi} = \frac{1}{2}\gamma B_k I\boldsymbol{\sigma}_k \psi = \frac{1}{2}\gamma I\mathbf{B}\psi, \quad (4.51)$$

where $\mathbf{B} = B_k \boldsymbol{\sigma}_k$. If we now decompose ψ into $\rho^{1/2}R$ we see that

$$\dot{\psi} \tilde{\psi} = \frac{1}{2}\dot{\rho} + \rho \dot{R} \tilde{R} = \frac{1}{2}\rho \gamma I\mathbf{B}. \quad (4.52)$$

The right-hand side is a bivector, so ρ must be constant. This is to be expected, as the evolution should be unitary. The dynamics reduces to

$$\dot{R} = \frac{1}{2}\gamma I\mathbf{B}R. \quad (4.53)$$

The quantum theory of a spin-1/2 particle in a magnetic field reduces to a simple rotor equation! This is very natural, if one thinks about the behaviour of particles in magnetic fields, and is the main justification for our approach.

Recovering a rotor equation explains the difficulty of the traditional analysis based on a pair of coupled equations for the components of $|\psi\rangle$. This approach fails to capture the fact that there is a rotor underlying the dynamics, and so carries along redundant degrees of freedom in the normalisation. Also, the separation of a rotor into a pair of components is far from natural. Two examples illustrate this. As a simple example, consider a constant field $\mathbf{B} = B_0 \boldsymbol{\sigma}_3$. The rotor equation integrates immediately to give

$$\psi(t) = e^{\gamma B_0 t I\boldsymbol{\sigma}_3/2} \psi_0. \quad (4.54)$$

The spin vector \mathbf{s} therefore just precesses about the 3 axis at a rate $\omega_0 = \gamma B_0$. Even this almost trivial result is substantially more complicated when following traditional methods.

4.3.2 Magnetic Resonance Imaging

A more interesting example is to include an oscillatory \mathbf{B} field $(B_1 \cos(\omega t), B_1 \sin(\omega t), 0)$ together with a constant field along the z -axis. This oscillatory field induces transitions

(spin-flips) between the up and down states, which differ in energy because of the constant component of the field. This is a very interesting system of great practical importance. It is the basis of magnetic resonance imaging and Rabi molecular beam spectroscopy.

To study this system we first write the \mathbf{B} field as

$$\begin{aligned} B_1(\cos(\omega t)\boldsymbol{\sigma}_1 + \sin(\omega t)\boldsymbol{\sigma}_2) + B_0\boldsymbol{\sigma}_3 &= e^{-\omega t I\boldsymbol{\sigma}_3} B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3 \\ &= e^{-\omega t I\boldsymbol{\sigma}_3/2} (B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3) e^{\omega t I\boldsymbol{\sigma}_3/2}. \end{aligned} \quad (4.55)$$

We now define

$$S = e^{-\omega t I\boldsymbol{\sigma}_3/2} \quad \text{and} \quad \mathbf{B}_c = B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3 \quad (4.56)$$

so that we can write $\mathbf{B} = S\mathbf{B}_c\tilde{S}$. The rotor equation can now be written

$$\tilde{S}\dot{\psi} = \frac{1}{2}\gamma I\mathbf{B}_c\tilde{S}\psi, \quad (4.57)$$

where we have pre-multiplied by \tilde{S} , and we continue to use ψ for the normalised rotor. Now noting that

$$\dot{\tilde{S}} = \frac{1}{2}\omega I\boldsymbol{\sigma}_3\tilde{S} \quad (4.58)$$

we see that

$$\frac{d}{dt}(\tilde{S}\psi) = \frac{1}{2}(\gamma I\mathbf{B}_c + \omega I\boldsymbol{\sigma}_3)\tilde{S}\psi. \quad (4.59)$$

It is now $\tilde{S}\psi$ that satisfies a rotor equation with a constant field. The solution is straightforward,

$$\tilde{S}\psi(t) = \exp\left(\frac{t}{2}(\gamma I\mathbf{B}_c + \omega I\boldsymbol{\sigma}_3)\right)\psi_0, \quad (4.60)$$

and we arrive at

$$\psi(t) = \exp\left(-\frac{\omega t}{2}I\boldsymbol{\sigma}_3\right) \exp\left(\frac{t}{2}((\omega_0 + \omega)I\boldsymbol{\sigma}_3 + \omega_1 I\boldsymbol{\sigma}_1)\right)\psi_0, \quad (4.61)$$

where $\omega_1 = \gamma B_1$. There are three separate frequencies in this solution, which contains a wealth of interesting physics. Needless to say, this derivation is a vast improvement over standard methods!

To complete our analysis we must relate our solution to the results of experiments. Suppose that at time $t = 0$ we switch on the oscillating field. The particle is initially in a spin-up state, so $\psi_0 = 1$, which also ensures that the state is normalised. The probability that at time t the particle is in the spin-down state is

$$P_{\downarrow} = |\langle \downarrow | \psi(t) \rangle|^2 \quad (4.62)$$

We therefore need to form the inner product

$$\langle \downarrow | \psi(t) \rangle \leftrightarrow \langle I\boldsymbol{\sigma}_2\psi \rangle_q = \langle I\boldsymbol{\sigma}_2\psi \rangle - I\boldsymbol{\sigma}_3\langle I\boldsymbol{\sigma}_2\psi I\boldsymbol{\sigma}_3 \rangle = \langle I\boldsymbol{\sigma}_2\psi \rangle - I\boldsymbol{\sigma}_3\langle I\boldsymbol{\sigma}_1\psi \rangle. \quad (4.63)$$

To find this inner product we write

$$\psi(t) = e^{-\omega t I \sigma_3 / 2} [\cos(\alpha t / 2) + I \hat{\mathbf{B}} \sin(\alpha t / 2)] \quad (4.64)$$

where

$$\hat{\mathbf{B}} = \frac{(\omega_0 + \omega) \sigma_3 + \omega_1 \sigma_1}{\alpha}, \quad \text{and} \quad \alpha = \sqrt{(\omega + \omega_0)^2 + \omega_1^2}. \quad (4.65)$$

The only term giving a contribution in the $I \sigma_1$ and $I \sigma_2$ planes is that in $\omega_1 I \sigma_1 / \alpha$. We therefore have

$$\langle I \sigma_2 \psi \rangle_q = \frac{\omega_1 \sin(\alpha t / 2)}{\alpha} e^{-\omega t I \sigma_3 / 2} I \sigma_3 \quad (4.66)$$

and the probability is immediately

$$P_{\downarrow} = \left(\frac{\omega_1 \sin(\alpha t / 2)}{\alpha} \right)^2. \quad (4.67)$$

The maximum value is at $\alpha t = \pi$, and the probability at this time is maximised by choosing α as small as possible. This is achieved by setting $\omega = -\omega_0 = -\gamma B_0$. This is the *spin resonance condition*.